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INTRODUCTION TO THE THEORY OF FOURIER INTEGRALS

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PREFACE

THE object of this book is to give a more systematic account of the elements of the theory of Fourier integrals than has hitherto been given. I have, however, not attempted to deal with a number of important topics of recent growth: Wiener's Tauberian theorems; applications to almost periodic functions, quasi-analytic functions, and integral functions; Stieltjes integrals; harmonic analysis in general; and Bochner's generalized integrals, and the theory for functions of several variables, of which an account is given in Bochner's book.

The reader requires only a general knowledge of analysis, though he will presumably be familiar with the elements of the theory of Fourier series. The book may be read as a sequel to my *Theory of Functions*.

A great variety of applications of Fourier integrals are to be found in the literature, often in the form of 'operators', and often in the works of authors who are evidently not specially interested in analysis. As exercises in the theory I have written out a few of these applications as it seemed to me that an analyst should. I have retained, as having a certain picturesqueness, some references to 'heat', 'radiation', and so forth; but the interest is purely analytical, and the reader need not know whether such things exist.

E. C. T.

NEW COLLEGE, OXFORD,
1937.

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I

CONVERGENCE AND SUMMABILITY

1.1. Fourier's formulae. The origin of the theory of Fourier integrals is to be found in Fourier's *Analytical Theory of Heat*.† Fourier's argument, which would not now be called a proof, is substantially as follows. Suppose that a function $f(x)$, of period $2\pi\lambda$, is represented by the Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{nx}{\lambda} + b_n\sin\frac{nx}{\lambda}\right).$$

The coefficients a_m, b_m are obtained formally by multiplying by $\cos(mx/\lambda)$ or $\sin(mx/\lambda)$, and integrating term-by-term over $(-\pi\lambda, \pi\lambda)$. This gives

$$a_m = \frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{mt}{\lambda}\,dt, \qquad b_m = \frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(t)\sin\frac{mt}{\lambda}\,dt,$$

and the formula may be written

$$f(x) = \frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{n(x-t)}{\lambda}\,dt.$$

Putting $n/\lambda = u$, $1/\lambda = \delta u$, and making $\lambda \to \infty$, the sum passes formally into an integral, and we obtain

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt. \tag{1.1.1}$$

This is *Fourier's integral formula*.

It may also be written in the form (analogous to that of the Fourier series)

$$f(x) = \int_{0}^{\infty}\{a(u)\cos xu + b(u)\sin xu\}\,du, \tag{1.1.2}$$

where

$$a(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos ut\,dt, \qquad b(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin ut\,dt. \tag{1.1.3}$$

If $f(t)$ is an even function, then

$$a(u) = \frac{2}{\pi}\int_{0}^{\infty} f(t)\cos ut\,dt,$$

---

† See list of books and monographs, pp. 370–1.

while $b(u)$ vanishes; and the formula becomes

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos xu\,du \int_{0}^{\infty} f(t)\cos ut\,dt. \tag{1.1.4}$$

This is *Fourier's cosine formula.* Similarly, if $f(x)$ is odd, $a(u)$ vanishes, and we obtain *Fourier's sine formula,*

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin xu\,du \int_{0}^{\infty} f(t)\sin ut\,dt. \tag{1.1.5}$$

We can also regard (1.1.1) as merely a combination of (1.1.4) and (1.1.5); for write

$$f(x) = \tfrac{1}{2}\{f(x)+f(-x)\}+\tfrac{1}{2}\{f(x)-f(-x)\} = g(x)+h(x),$$

so that $g(x)$ is even and $h(x)$ is odd. Then

$$\begin{aligned}&\int_{\infty}^{\infty} f(t)\cos u(x-t)\,dt\\ &\qquad = 2\cos ux\int_{0}^{\infty} g(t)\cos ut\,dt + 2\sin ux\int_{0}^{\infty} h(t)\sin ut\,dt,\end{aligned}$$

and (1.1.1) gives

$$\begin{aligned}&g(x)+h(x)\\ &\quad = \frac{2}{\pi}\int_{0}^{\infty} \cos xu\,du \int_{0}^{\infty} g(t)\cos ut\,dt + \frac{2}{\pi}\int_{0}^{\infty} \sin xu\,du \int_{0}^{\infty} h(t)\sin ut\,dt,\end{aligned}$$

i.e. the cosine formula for $g(x)$ added to the sine formula for $h(x)$.

The above formulae were discovered independently by Cauchy† in his researches on the propagation of waves. The formal basis given by Cauchy is as follows. The right-hand side of (1.1.1) is, formally, the limit as $\delta \to 0$ of

$$\begin{aligned}\frac{1}{\pi}\int_{0}^{\infty} e^{-\delta u}\,du \int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} e^{-\delta u}\cos u(x-t)\,du\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\delta}{\delta^2+(x-t)^2}\,dt.\end{aligned}$$

The factor multiplying $f(t)$ tends to 0 except when $t = x$. We should therefore expect the value of the integral to be unaltered if we replace $f(t)$ by $f(x)$; and this would give

---

† Cauchy (1), (2); see list of references at the end of the book.

$$\frac{f(x)}{\pi}\int_{-\infty}^{\infty}\frac{\delta}{\delta^2+(x-t)^2}\,dt = f(x),$$

again verifying (1.1.1).

Another equivalent formula, given by Cauchy, is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixu}\,du \int_{-\infty}^{\infty} f(t)e^{iut}\,dt. \tag{1.1.6}$$

Putting $f(x) = g(x)+h(x)$, where g is even and h odd, as before,

$$\int_{-\infty}^{\infty} f(t)e^{iut}\,dt = 2\int_{0}^{\infty} g(t)\cos ut\,dt + 2i\int_{0}^{\infty} h(t)\sin ut\,dt,$$

and the right-hand side of (1.1.6) is

$$\frac{2}{\pi}\int_{0}^{\infty}\cos xu\,du\int_{0}^{\infty} g(t)\cos ut\,dt + \frac{2}{\pi}\int_{0}^{\infty}\sin xu\,du\int_{0}^{\infty} h(t)\sin ut\,dt$$
$$= g(x)+h(x) = f(x).$$

We shall call (1.1.6) the exponential form of Fourier's formula.

A formula of a slightly different type is obtained by expressing the outer integral in (1.1.1) as the limit of an integral over $(0,\lambda)$, and inverting the order of integration. The result is

$$f(x) = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt. \tag{1.1.7}$$

The same result may be obtained in the same way from (1.1.6). This formula is known as *Fourier's single-integral formula*.

1.2. Fourier transforms. It was pointed out by Cauchy that these formulae lead to reciprocal relations between pairs of functions. If we write

$$F_c(u) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} f(t)\cos ut\,dt, \tag{1.2.1}$$

then (1.1.4) is

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} F_c(u)\cos xu\,du, \tag{1.2.2}$$

and the relation between $f(x)$ and $F_c(x)$ is reciprocal. Such functions were called by Cauchy reciprocal functions of the first kind. We shall call functions so related *Fourier cosine transforms* of each other. Thus

$$e^{-x}, \qquad \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{1+x^2}$$

are a pair of Fourier cosine transforms.

Similarly, from Fourier's sine formula, we obtain

$$F_s(u) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(t)\sin ut\,dt, \tag{1.2.3}$$

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F_s(u)\sin xu\,du. \tag{1.2.4}$$

These were called by Cauchy reciprocal functions of the second kind. We shall call them *Fourier sine transforms* of each other. Thus e^{-x}, $\sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{1+x^2}$ are Fourier sine transforms.

The formula (1.1.6) leads similarly to the unsymmetrical formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^\infty f(t)e^{iut}\,dt, \tag{1.2.5}$$

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^\infty F(u)e^{-ixu}\,du. \tag{1.2.6}$$

We shall call such functions simply *Fourier transforms* of each other. Thus

$$f(x) = e^{-|x|}, \qquad F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{1+x^2}$$

are Fourier transforms of each other.

If $f(x)$ is even, $F(x) = F_c(x)$; if $f(x)$ is odd, $F(x) = iF_s(x)$.

1.3. Generalized Fourier integrals. The existence of the integral defining $F(u)$ implies a certain restriction on $f(x)$ at infinity. Even if $F(u)$ does not exist, the functions

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty f(t)e^{iwt}\,dt, \tag{1.3.1}$$

$$F_-(w) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^0 f(t)e^{iwt}\,dt, \tag{1.3.2}$$

where $w = u+iv$, may exist, the former for sufficiently large positive v, the latter for sufficiently large negative v. For

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty f(t)e^{-vt}e^{iut}\,dt, \tag{1.3.3}$$

so that $F_+(w)$ is the transform of the function equal to $f(t)e^{-vt}$ for $t > 0$, and to 0 for $t < 0$. The formula reciprocal to (1.3.3) is

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F_+(u+iv)e^{-ixu}\,du = \begin{cases} f(x)e^{-vx} & (x > 0) \\ 0 & (x < 0), \end{cases}$$

or
$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F_+(u+iv)e^{-ix(u+iv)}\,du = \begin{cases} f(x) & (x > 0) \\ 0 & (x < 0). \end{cases}$$

There is a similar formula involving F_-. Adding, we may write

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{ia-\infty}^{ia+\infty} F_+(w)e^{-ixw}\,dw + \frac{1}{\sqrt{(2\pi)}} \int_{ib-\infty}^{ib+\infty} F_-(w)e^{-ixw}\,dw, \tag{1.3.4}$$

where a is a sufficiently large positive number, b a sufficiently large negative number.

For example, if $f(x) = e^x$, then

$$F_+(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{1}{1+iw}, \qquad F_-(w) = \frac{1}{\sqrt{(2\pi)}}\frac{1}{1+iw}.$$

In this case (1.3.4) is at once verified by the calculus of residues.

In this form Fourier's integral formula may be applied to a periodic function. Let $f(x)$ have the period 2π. Then for $v > 0$

$$F_+(w) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty f(x)e^{ixw}\,dx = \frac{1}{\sqrt{(2\pi)}} \sum_{n=0}^{\infty} \int_{2n\pi}^{2(n+1)\pi} f(x)e^{ixw}\,dx$$
$$= \frac{1}{\sqrt{(2\pi)}} \sum_{n=0}^{\infty} \int_0^{2\pi} f(\xi)e^{i(\xi+2n\pi)w}\,d\xi = \frac{1}{\sqrt{(2\pi)}} \int_0^{2\pi} f(\xi)\frac{e^{i\xi w}}{1-e^{2\pi iw}}\,d\xi$$
$$= \frac{1}{\sqrt{(2\pi)}}\frac{\phi(w)}{1-e^{2\pi iw}},$$

where
$$\phi(w) = \int_0^{2\pi} f(\xi)e^{i\xi w}\,d\xi.$$

Similarly, $$F_-(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{\phi(w)}{1-e^{2\pi i w}} \quad (v < 0).$$

The reciprocal formula is therefore

$$f(x) = \frac{1}{2\pi}\int\limits_{ia-\infty}^{ia+\infty} \frac{\phi(w)}{1-e^{2\pi i w}} e^{-ixw}\,dw - \frac{1}{2\pi}\int\limits_{ib-\infty}^{ib+\infty} \frac{\phi(w)}{1-e^{2\pi i w}} e^{-ixw}\,dw.$$

Here $\phi(w)$ is an integral function. If it behaves at infinity so that we can evaluate the right-hand side by the calculus of residues in the obvious way, we obtain

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \phi(n)e^{-inx}.$$

We have thus returned to the Fourier series for $f(x)$.

1.4. The formulae of Laplace. The formula

$$\phi(s) = \int\limits_0^{\infty} f(x)e^{-sx}\,dx \tag{1.4.1}$$

is known as Laplace's integral. If $f(x)$ is the given function, $\phi(s)$ is in general analytic for $\mathbf{R}(s) > 0$. The reciprocal formula is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \phi(s)e^{sx}\,ds = \begin{cases} f(x) & (x > 0) \\ 0 & (x < 0). \end{cases} \tag{1.4.2}$$

From a formal point of view the formulae are a particular case of those of § 1.2, as is seen on putting $s = \sigma + it$.

As a still more special case we obtain a reciprocity between two analytic functions. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and suppose that the integral (1.4.1) can be evaluated by term-by-term integration. Then

$$\phi(s) = \sum_{n=0}^{\infty} a_n \int\limits_0^{\infty} e^{-sx}x^n\,dx = \sum_{n=0}^{\infty} \frac{n!\,a_n}{s^{n+1}},$$

or $$\frac{1}{s}\phi\left(\frac{1}{s}\right) = \sum_{n=0}^{\infty} n!\,a_n s^n.$$

If $f(x)$ is suitably restricted, $\phi(s)$ will be an analytic function regular in the neighbourhood of $s = \infty$; and, if C is a closed curve surrounding the origin, but lying sufficiently far from it,

$$\frac{1}{2\pi i}\int\limits_C \phi(w)e^{zw}\,dw = \frac{1}{2\pi i}\sum_{n=0}^{\infty} n!\,a_n \int\limits_C \frac{e^{zw}}{w^{n+1}}\,dw$$

$$= \sum_{n=0}^{\infty} a_n z^n = f(z). \tag{1.4.3}$$

The function $f(z)$ may therefore be represented as a trigonometrical integral, but now along a closed curve.

1.5. The formulae of Mellin. Still another pair of formulae embodying the same formal idea is given by

$$\mathfrak{F}(s) = \int\limits_0^{\infty} f(x)x^{s-1}\,dx, \tag{1.5.1}$$

$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathfrak{F}(s)x^{-s}\,ds. \tag{1.5.2}$$

The idea of such a reciprocity occurs in Riemann's famous memoir† on prime numbers. It was formulated explicitly by Cahen,‡ and the first accurate discussion was given by Mellin.|| We shall call the formulae *Mellin's inversion formulae.*

These formulae arise naturally in the theory of Dirichlet series in the following way. The particular case

$$\Gamma(s) = \int\limits_0^{\infty} e^{-x}x^{s-1}\,dx, \qquad e^{-x} = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\,ds \quad (c > 0)$$

is well known. Now let $\phi(s)$ be a function expressible as a Dirichlet series,

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Then we have formally

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\Gamma(s)}\int\limits_0^{\infty} e^{-nx}x^{s-1}\,dx = \frac{1}{\Gamma(s)}\int\limits_0^{\infty} f(x)x^{s-1}\,dx,$$

where
$$f(x) = \sum_{n=0}^{\infty} a_n e^{-nx};$$

---

† Riemann (1). ‡ Cahen (1). || Mellin (1), (2).

and reciprocally

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi(s)\Gamma(s)x^{-s}\,ds = \sum_{n=1}^{\infty}\frac{a_n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)(nx)^{-s}\,ds$$

$$= \sum_{n=1}^{\infty}a_n e^{-nx} = f(x).$$

The forms (1.5.1), (1.5.2) are obtained by putting $\phi(s)\Gamma(s) = \mathfrak{F}(s)$.

Mellin's formulae may also be obtained by a substitution from the exponential form of Fourier's formula. In fact, putting $x = e^{\xi}$ and $s = c+it$, (1.5.1) becomes

$$\mathfrak{F}(c+it) = \int_{-\infty}^{\infty} f(e^{\xi})e^{\xi(c+it)}\,d\xi,$$

and (1.5.2) becomes

$$f(e^{\xi}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathfrak{F}(c+it)e^{-\xi(c+it)}\,dt.$$

The functions $\sqrt{(2\pi)}e^{c\xi}f(e^{\xi}),\quad \mathfrak{F}(c+it)$

are thus Fourier transforms of each other.

Suppose that, in Mellin's formulae, the function $f(x)$ is analytic at the origin and in a region containing the positive real axis. Consider the integral

$$\int_{\Gamma} f(z)(-z)^{s-1}\,dz,$$

where Γ is a loop coming from infinity on the positive real axis, encircling the origin in the positive direction, and returning to infinity. We define $(-z)^{s-1}$ as $e^{(s-1)\log(-z)}$, where $\log(-z)$ is real on the negative real axis.

Suppose Γ compressed into the real axis on both sides. The part of the integral above the real axis gives

$$-\int_{0}^{\infty} f(x)e^{(s-1)(\log x-i\pi)}\,dx = e^{-is\pi}\int_{0}^{\infty} f(x)x^{s-1}\,dx,$$

and that below the real axis gives

$$\int_{0}^{\infty} f(x)e^{(s-1)(\log x+i\pi)}\,dx = -e^{is\pi}\int_{0}^{\infty} f(x)x^{s-1}\,dx.$$

Hence $$\int_{\Gamma} f(z)(-z)^{s-1}\,dz = -2i\sin s\pi\,\mathfrak{F}(s).$$

Let $$\pi\chi(s) = \mathfrak{F}(s)\sin s\pi.$$

Then we obtain the reciprocal formulae

$$\chi(s) = -\frac{1}{2i\pi}\int\limits_{\Gamma} f(z)(-z)^{s-1}\,dz, \tag{1.5.3}$$

$$f(z) = \frac{1}{2i}\int\limits_{c-i\infty}^{c+i\infty} \frac{\chi(s)z^{-s}}{\sin\pi s}\,ds. \tag{1.5.4}$$

A simple example is $f(z) = e^{-z}$, $\chi(s) = 1/\Gamma(1-s)$. Such formulae have important applications in the theory of functions of a complex variable,† but we cannot consider them further here.

1.6. For the early history of the Fourier-Cauchy formulae we may refer to the article by Burkhardt in the *Encyklopädie.*

The theorems of this chapter are in the main analogous to classical theorems in the theory of Fourier series. We do not actually assume a knowledge of the theory of Fourier series, though the reader will presumably be familiar with it. Almost all theorems on Fourier series have some sort of analogue for integrals. In some cases the theorems are so similar that the extension from series to integrals is hardly worth making. In other cases there are new points of interest in the integral case, which is even sometimes the simpler.

1.7. Notation.

We use

$$\int\limits_0^\infty f(x)\,dx$$

to denote the Lebesgue integral of $f(x)$ over $(0,\infty)$ in the strict sense, implying that the integral is absolutely convergent, i.e. that

$$\int\limits_0^\infty |f(x)|\,dx$$

also exists. If $f(x)$ is integrable over $(0, X)$ for every X, and

$$\lim_{X\to\infty}\int\limits_0^X f(x)\,dx$$

exists, we denote the limit by

$$\int\limits_0^{\to\infty} f(x)\,dx.$$

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† Carlson (1).

Such an integral is known as a Cauchy integral. A similar notation is used in the case of other limits. Thus

$$\int_{\to 0}^{1} f(x)\,dx$$

denotes the limit of
$$\int_{\delta}^{1} f(x)\,dx$$

as $\delta \to 0$ through positive values.

In 'formal' analysis we use $\int_{0}^{\infty} f(x)\,dx$ to denote that the integral exists in some sense or other. There is generally little risk of confusion between this and the Lebesgue sense.

We say that $f(x)$ belongs to, or is, $L^{p}(a,b)$ if $f(x)$ is measurable and

$$\int_{a}^{b} |f(x)|^{p}\,dx < \infty.$$

We write L for L^{1}.

By
$$\underset{X\to\infty}{\text{l.i.m.}} \int_{0}^{X} f(x,\alpha)\,dx$$

(limit in mean) we denote a function $\phi(\alpha)$ such that

$$\lim_{X\to\infty} \int_{a}^{b} \left|\phi(\alpha) - \int_{0}^{X} f(x,\alpha)\,dx\right|^{p} d\alpha = 0,$$

a, b, and p having prescribed values.

As complex variables we use

$$z = x+iy, \quad w = u+iv, \quad s = \sigma+it, \quad \zeta = \xi+i\eta.$$

If $f(x)$ is a given function, we denote by

$$F(x),\ F_{c}(x),\ F_{s}(x),\ F_{+}(w),\ F_{-}(w),\ \mathfrak{F}(s),$$

the functions defined in (1.2.5), (1.2.1), (1.2.3), (1.3.1), (1.3.2), (1.5.1) respectively. In each case it is assumed that the integral referred to exists in some sense or other. The ambiguity of the expression 'a Fourier transform', arising from the asymmetry of the formulae (1.2.5), (1.2.6), is avoided by standardizing the use of small and capital letters as in these formulae.

Similarly with other letters (g, G, G_{+}, G_{-}, etc.).

We denote by A an absolute constant, not necessarily the same one at each occurrence; K is used in a similar way for a constant depending on the data of the problem in hand.

We say that the convergence of a sequence $f_n(x)$ to a limit $f(x)$ is bounded if $|f_n(x)| \leqslant K$ for all n and x; and that it is dominated if $|f_n(x)| \leqslant \phi(x)$, where $\phi(x)$ is L over a prescribed set. It is known† that

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int f(x)\,dx$$

if the convergence is bounded or dominated.

1.8. Fundamental theorems. The theorem of Riemann-Lebesgue is fundamental in the theory of Fourier integrals, as it is in the theory of Fourier series. We shall state it as follows.

THEOREM 1. *Let* $f(x)$ *belong to* $L(-\infty,\infty)$. *Then the integrals*

$$\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx, \qquad \int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx, \tag{1.8.1}$$

tend to zero as $\lambda \to \infty$.

Consider the cosine integral. Let ϵ be a given positive number. Then we can choose X so large that

$$\int_{X}^{\infty} |f(x)|\,dx < \epsilon, \qquad \int_{-\infty}^{-X} |f(x)|\,dx < \epsilon.$$

Hence $$\left|\int_{X}^{\infty} f(x)\cos\lambda x\,dx\right| < \epsilon, \qquad \left|\int_{-\infty}^{-X} f(x)\cos\lambda x\,dx\right| < \epsilon$$

for all values of λ.

Next, we can define a function $\phi(x)$, absolutely continuous in the interval $(-X, X)$, such that

$$\int_{-X}^{X} |f(x)-\phi(x)|\,dx < \epsilon.$$

Then $$\left|\int_{-X}^{X} \{f(x)-\phi(x)\}\cos\lambda x\,dx\right| < \epsilon$$

for all values of λ. Finally

$$\int_{-X}^{X} \phi(x)\cos\lambda x\,dx$$

$$= \frac{\phi(X)\sin\lambda X}{\lambda} + \frac{\phi(-X)\sin\lambda X}{\lambda} - \frac{1}{\lambda}\int_{-X}^{X} \phi'(x)\sin\lambda x\,dx,$$

and (for a fixed X) we can choose λ_0 so large that the modulus of this is less than ϵ for $\lambda > \lambda_0$. Then

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† Titchmarsh, *Theory of Functions*, §§ 10.5, 10.8.

$$\left|\int\limits_{-\infty}^{\infty} f(x)\cos\lambda x\,dx\right| < 4\epsilon \quad (\lambda > \lambda_0).$$

This proves the theorem for the cosine integral; a similar proof applies to the sine integral.

Theorem 2. *Let $f(x)$ belong to $L(-\infty,\infty)$. Then a necessary and sufficient condition that*

$$\frac{1}{\pi}\int\limits_{0}^{\to\infty} du \int\limits_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt = a \tag{1.8.2}$$

is that, for any fixed δ,

$$\lim_{\lambda\to\infty}\int\limits_{0}^{\delta} \{f(x+y)+f(x-y)-2a\}\frac{\sin\lambda y}{y}\,dy = 0. \tag{1.8.3}$$

Since $|f(t)\cos u(x-t)| \leqslant |f(t)|$, the integral

$$\int\limits_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt$$

converges uniformly with respect to u over any finite interval. Hence

$$\int\limits_{0}^{\lambda} du \int\limits_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt = \int\limits_{-\infty}^{\infty} f(t)\,dt \int\limits_{0}^{\lambda} \cos u(x-t)\,du$$

$$= \int\limits_{-\infty}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt.$$

Since $f(t)/(x-t)$ is integrable over $(-\infty, x-\delta)$ and $(x+\delta,\infty)$, it follows from the Riemann-Lebesgue theorem that, for a fixed δ,

$$\lim_{\lambda\to\infty}\int\limits_{-\infty}^{x-\delta} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = 0, \qquad \lim_{\lambda\to\infty}\int\limits_{x+\delta}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = 0.$$

Also

$$\int\limits_{x-\delta}^{x+\delta} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = \int\limits_{0}^{\delta} \{f(x+y)+f(x-y)\}\frac{\sin\lambda y}{y}\,dy,$$

and

$$\lim_{\lambda\to\infty}\int\limits_{0}^{\delta} 2a\frac{\sin\lambda y}{y}\,dy = \lim_{\lambda\to\infty} 2a\int\limits_{0}^{\lambda\delta}\frac{\sin v}{v}\,dv = 2a\int\limits_{0}^{\to\infty}\frac{\sin v}{v}\,dv = a\pi.$$

These equations together show that (1.8.2) and (1.8.3) are equivalent.

1.9. We are now in a position to extend all the ordinary convergence tests for Fourier series to Fourier integrals. We shall, however, content ourselves with proving the two following theorems, corresponding to the tests of Jordan and Dini respectively.

THEOREM 3. *Let $f(t)$ belong to $L(-\infty,\infty)$. If $f(t)$ is of bounded variation in an interval including the point x, then*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{1}{\pi}\int\limits_{0}^{\to\infty} du \int\limits_{-\infty}^{\infty} f(t)\cos u(x-t)\, dt. \qquad (1.9.1)$$

If $f(t)$ is continuous and of bounded variation in an interval (a,b), then

$$f(x) = \frac{1}{\pi}\int\limits_{0}^{\to\infty} du \int\limits_{-\infty}^{\infty} f(t)\cos u(x-t)\, dt, \qquad (1.9.2)$$

the integral converging uniformly in any interval interior to (a,b).

Let $\qquad \psi(y) = f(x+y)+f(x-y)-f(x+0)-f(x-0).$

Then $\psi(y)$ is of bounded variation over $(0,\delta)$, if δ is small enough, and $\psi(y) \to 0$ as $y \to 0$. We may therefore write

$$\psi(y) = \psi_1(y)-\psi_2(y),$$

where $\psi_1(y)$ and $\psi_2(y)$ are positive non-decreasing bounded functions in $(0,\delta)$, which tend to 0 as $y \to 0$.

Given any positive number ϵ, there is a number η such that $\psi_1(y) \leqslant \epsilon$ for $y \leqslant \eta$. Let

$$\int\limits_{0}^{\delta} \psi_1(y)\frac{\sin\lambda y}{y}\,dy = \int\limits_{0}^{\eta} \psi_1(y)\frac{\sin\lambda y}{y}\,dy + \int\limits_{\eta}^{\delta} \psi_1(y)\frac{\sin\lambda y}{y}\,dy.$$

By the second mean-value theorem, the first part is equal to

$$\psi_1(\eta)\int\limits_{\xi}^{\eta} \frac{\sin\lambda y}{y}\,dy = \psi_1(\eta)\int\limits_{\lambda\xi}^{\lambda\eta} \frac{\sin v}{v}\,dv \quad (0<\xi<\eta),$$

and the last integral is bounded for all λ and ξ. Hence

$$\left|\int\limits_{0}^{\eta} \psi_1(y)\frac{\sin\lambda y}{y}\,dy\right| < A\epsilon,$$

for all values of λ. Having fixed η, $\psi_1(y)/y$ is integrable over (η,δ), so that

$$\lim_{\lambda\to\infty}\int\limits_{\eta}^{\delta} \psi_1(y)\frac{\sin\lambda y}{y}\,dy = 0.$$

Since ϵ is arbitrary, it follows that

$$\lim_{\lambda\to\infty}\int\limits_0^\delta \psi_1(y)\frac{\sin\lambda y}{y}\,dy = 0.$$

Similarly, the integral involving $\psi_2(y)$ tends to 0. This proves the first clause of the theorem.

If $f(x)$ is continuous in (a,b), $\frac{1}{2}\{f(x+0)+f(x-0)\} = f(x)$; and, the function being uniformly continuous in any interval interior to (a,b), the conditions used in the proof hold uniformly, and so the convergence is uniform.

Theorem 4. *Let $f(t)$ belong to $L(-\infty,\infty)$. Then, for a given x, (1.9.2) is true if*

$$\int\limits_0^\delta \left|\frac{f(x+y)+f(x-y)-2f(x)}{y}\right| dy \tag{1.9.3}$$

exists for some positive δ; in particular it holds if $f(x)$ is differentiable at the point x.

This follows at once from Theorems 1 and 2, with $a = f(x)$. If $f(x)$ is differentiable, the integrand in (1.9.3) is bounded, so that the condition is plainly satisfied.

Theorem 5.[†] *Let $f(t)/(1+|t|)$ belong to $L(-\infty,\infty)$; let*

$$a_1(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(y)\frac{\sin xy}{y}\,dy, \tag{1.9.4}$$

$$b_1(x) = \frac{1}{\pi}\int\limits_{-1}^{1} f(y)\frac{1-\cos xy}{y}\,dy - \frac{1}{\pi}\left(\int\limits_{-\infty}^{-1} + \int\limits_1^{\infty}\right) f(y)\frac{\cos xy}{y}\,dy \tag{1.9.5}$$

be absolutely continuous over any finite interval $0 < \delta \leqslant x \leqslant \Delta$, and let $a(x)$, $b(x)$ be their respective derivatives. Let $f(t)$ satisfy the conditions of Theorem 3 or Theorem 4 in the neighbourhood of $t = x$. Then

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \int\limits_{\to 0}^{\to\infty} \{a(u)\cos xu + b(u)\sin xu\}\,du.$$

Suppose first that $f(x) = 0$ for $|x| \geqslant 1$. Then

$$\frac{1}{\pi}\int\limits_0^x d\xi \int\limits_{-1}^1 f(y)\cos\xi y\,dy = \frac{1}{\pi}\int\limits_{-1}^1 f(y)\frac{\sin xy}{y}\,dy = a_1(x),$$

[†] Hahn (2).

so that
$$a(x) = \frac{1}{\pi}\int_{-1}^{1} f(y)\cos xy\,dy$$
almost everywhere. Similarly,
$$b(x) = \frac{1}{\pi}\int_{-1}^{1} f(y)\sin xy\,dy$$
almost everywhere. The result then follows from Theorem 3 or Theorem 4.

Suppose next that $f(x) = 0$ for $|x| < 1$. Then $f(x)/x$ belongs to $L(-\infty,\infty)$. Hence, by Theorem 3 or 4,
$$\frac{1}{2x}\{f(x+0)+f(x-0)\} = \frac{1}{\pi}\int_{0}^{\to\infty}\{-b_1(u)\cos xu + a_1(u)\sin xu\}\,du.$$
Now
$$\int_{0}^{\to\infty} b_1(u)\cos xu\,du = \left[b_1(u)\frac{\sin xu}{x}\right]_0^{\infty} - \frac{1}{x}\int_{\to 0}^{\to\infty} b(u)\sin xu\,du$$
$$= -\frac{1}{x}\int_{\to 0}^{\to\infty} b(u)\sin xu\,du,$$
since $b_1(u) \to 0$ as $u \to \infty$.

Also
$$\int_{0}^{\to\infty} a_1(u)\sin xu\,du = \left[-a_1(u)\frac{\cos xu}{x}\right]_0^{\infty} + \frac{1}{x}\int_{\to 0}^{\to\infty} a(u)\cos xu\,du$$
$$= \frac{1}{x}\int_{\to 0}^{\to\infty} a(u)\cos xu\,du,$$
since $a_1(u)$ tends to 0 as $u \to 0$ or $u \to \infty$. The result in this case thus follows.

The general result now follows by adding functions of the two classes considered.

1.10. Monotonic functions.[†] The next theorem is based on the fact that, even if $\int_0^\infty f(t)\,dt$ does not exist, the integrals
$$\int_{0}^{\to\infty} f(t)\cos ut\,dt, \qquad \int_{0}^{\to\infty} f(t)\sin ut\,dt$$
exist for $u > 0$ provided that $f(t) \to 0$ steadily as $t \to \infty$. Here it

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[†] Pringsheim (1).

seems slightly more convenient to take the cosine and sine integrals separately.

THEOREM 6. *Let $f(t)$ be non-increasing over $(0,\infty)$, integrable over any finite interval beginning at 0, and let $f(t)\to 0$ as $t\to\infty$. Then for any positive x*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{2}{\pi}\int\limits_{\to 0}^{\to\infty} \cos xu\,du \int\limits_{0}^{\to\infty} f(t)\cos ut\,dt.$$

We have, by the second mean-value theorem,

$$\left|\int\limits_{T}^{T'} f(t)\cos ut\,dt\right| = \left|f(T+0)\int\limits_{T}^{T''} \cos ut\,dt\right| \leqslant \frac{2f(T+0)}{u}.$$

Hence the t-integral converges uniformly with respect to u over $0<\lambda\leqslant u\leqslant\mu$. Hence

$$\int\limits_{\lambda}^{\mu} \cos xu\,du \int\limits_{0}^{\to\infty} f(t)\cos ut\,dt = \int\limits_{0}^{\to\infty} f(t)\,dt \int\limits_{\lambda}^{\mu} \cos xu\cos ut\,du$$

$$= \frac{1}{2}\int\limits_{0}^{\to\infty} f(t)\left\{\frac{\sin\mu(x-t)}{x-t}-\frac{\sin\lambda(x-t)}{x-t}+\frac{\sin\mu(x+t)}{x+t}-\frac{\sin\lambda(x+t)}{x+t}\right\}dt.$$

Now

$$\left|\int\limits_{T}^{\to\infty} f(t)\frac{\sin\mu(x-t)}{x-t}\,dt\right| = \left|f(T+0)\int\limits_{T}^{T'} \frac{\sin\mu(x-t)}{x-t}\,dt\right| < Af(T+0),$$

and similarly for the integrals involving λ and $x+t$. We can therefore choose T so large that

$$\left|\int\limits_{T}^{\to\infty} f(t)\left\{\frac{\sin\mu(x-t)}{x-t}-\ldots\right\}dt\right| < \epsilon$$

for $T>T_0(\epsilon)$, for all values of λ and μ. Having fixed $T>x$,

$$\lim_{\mu\to\infty}\int\limits_{0}^{T} f(t)\frac{\sin\mu(x-t)}{x-t}\,dt = \tfrac{1}{2}\pi\{f(x+0)+f(x-0)\}$$

by the analysis of Theorem 3, and

$$\lim_{\mu\to\infty}\int\limits_{0}^{T} f(t)\frac{\sin\mu(x+t)}{x+t}\,dt = 0$$

by the Riemann-Lebesgue theorem. Also

$$\left|\int_0^T f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt\right| \leqslant \lambda\int_0^T f(t)\,dt \to 0$$

as $\lambda \to 0$; and similarly for the remaining part.

Theorem 7. *If $f(t)$ satisfies the conditions of Theorem* 6, *then for any positive x*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{2}{\pi}\int_0^{\to\infty}\sin xu\,du\int_0^{\to\infty} f(t)\sin ut\,dt.$$

As $u \to 0$

$$\left|\int_1^{\to\infty} f(t)\sin ut\,dt\right| = \left|f(1+0)\int_1^T \sin ut\,dt\right|$$

$$= f(1+0)\left|\frac{\cos u-\cos uT}{u}\right| \leqslant \frac{2f(1+0)}{u},$$

and

$$\int_0^1 f(t)\sin ut\,dt = O(1).$$

Hence the u-integral is absolutely convergent at the lower limit. Apart from this, the proof is the same as that of Theorem 6.

Fourier's formulae may be established under still more general conditions by adding a function of the type of Theorem 3 to one of the type of Theorem 6. The results of this process are sufficiently obvious.

1.11. Functions containing a periodic factor.†

Theorem 8. *Let* $f(t) = g(t)\cos at$ $(a > 0)$, *where* $g(t)$ *is non-increasing, integrable over* $(0, 1)$, *and* $g(t) \to 0$ *as* $t \to \infty$. *Then for any positive x*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{2}{\pi}\left(\int_0^{\to a}+\int_{\to a}^{\to\infty}\right)\cos xu\,du\int_0^{\to\infty} f(t)\cos ut\,dt.$$

The inner integral is

$$\int_0^{\to\infty} g(t)\cos at\cos ut\,dt,$$

which is uniformly convergent over any finite interval not including or ending at $u = a$. We may therefore invert the integral

$$\int_0^{a-\delta}\cos xu\,du\int_0^{\to\infty} g(t)\cos at\cos ut\,dt$$

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† Pringsheim (1).

for every $\delta > 0$. To prove that

$$\int_0^{\to a} \cos xu\,du \int_0^{\to\infty} g(t)\cos at\cos ut\,dt = \int_0^{\to\infty} \dots \int_0^{\to a} \dots$$

it is therefore sufficient to prove that

$$\lim_{\delta\to 0} \int_0^{\to\infty} g(t)\cos at\,dt \int_{a-\delta}^{a} \cos xu\cos ut\,du = 0.$$

This is clearly true for the part $0 \leqslant t \leqslant T$, with any finite T. It is therefore sufficient to prove that

$$\lim_{\delta\to 0} \int_T^{\to\infty} g(t)\cos at\,dt \int_{a-\delta}^{a} \cos xu\cos ut\,du = 0,$$

or

$$\lim_{\delta\to 0} \int_T^{\to\infty} g(t)\cos at\left\{\frac{\sin a(x-t)-\sin(a-\delta)(x-t)}{x-t}+\right.$$
$$\left.+\frac{\sin a(x+t)-\sin(a-\delta)(x+t)}{x+t}\right\}dt = 0.$$

Clearly, if $T > x$,

$$\int_T^{\to\infty} g(t)\cos at\{\sin a(x-t)-\sin(a-\delta)(x-t)\}\left(\frac{1}{x-t}+\frac{1}{t}\right)dt \to 0,$$

since this integral converges uniformly with respect to δ. Similarly for the integral involving $x+t$. Hence it is sufficient to consider

$$\int_T^{\to\infty} \frac{g(t)\cos at}{t}\{\sin a(x+t)-\sin(a-\delta)(x+t)-$$
$$-\sin a(x-t)+\sin(a-\delta)(x-t)\}\,dt$$
$$= 2\int_T^{\to\infty} \frac{g(t)\cos at}{t}\{\cos ax\sin at-\cos(a-\delta)x\sin(a-\delta)t\}\,dt.$$

Now

$$2\int_T^{\to\infty} \frac{g(t)}{t}\cos at\sin(a-\delta)t\,dt = \int_T^{\to\infty} \frac{g(t)}{t}\{\sin(2a-\delta)t-\sin\delta t\}\,dt,$$

which converges uniformly with respect to δ as $\delta \to 0$, since

$$\int_{T_1}^{T_2} \frac{g(t)}{t}\sin\delta t = O\left\{g(T_1)\int_{T_1}^{T_1'} \frac{\sin\delta t}{t}\,dt\right\} = O\{g(T_1)\}:$$

Hence $$\int_T^{\to\infty}\frac{g(t)}{t}\cos at\sin(a-\delta)t\,dt\to\int_T^{\to\infty}\frac{g(t)}{t}\cos at\sin at\,dt,$$
and the result follows.

A similar argument applies to the integral over $(a+\delta,\lambda)$. It therefore follows that
$$\left(\int_0^{\to a}+\int_{\to a}^{\lambda}\right)\cos xu\,du\int_0^{\to\infty}f(t)\cos ut\,dt$$
$$=\frac{1}{2}\int_0^{\to\infty}f(t)\left\{\frac{\sin\lambda(x-t)}{x-t}+\frac{\sin\lambda(x+t)}{x+t}\right\}dt.$$
Finally
$$\int_T^{T'}\cos at\frac{\sin\lambda(x-t)}{x-t}\,dt=\int_{T-x}^{T'-x}\cos a(x+y)\frac{\sin\lambda y}{y}\,dy$$
$$=\cos ax\int_{T-x}^{T'-x}\frac{\cos ay\sin\lambda y}{y}\,dy-\sin ax\int_{T-x}^{T'-x}\frac{\sin ay\sin\lambda y}{y}\,dy$$
is bounded for fixed a and x, $T>2x$, $\lambda>2a$. Hence, by the second mean-value theorem,
$$\int_T^{\to\infty}g(t)\cos at\frac{\sin\lambda(x-t)}{x-t}\,dt=O\{g(T)\},$$
and the proof concludes as in Theorem 6.

Theorem 9. *Let $f(t)=g(t)\sin at$ $(a>0)$, where $g(t)$ satisfies the same conditions as in Theorem 8. Then for any positive x*
$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=\frac{2}{\pi}\lim_{\delta\to0}\left(\int_0^{a-\delta}+\int_{a+\delta}^{\to\infty}\right)\cos xu\,du\int_0^{\to\infty}f(t)\cos ut\,dt.$$
If, in addition, $\int^{\infty}\frac{g(x)}{x}\,dx$ *exists, then*
$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=\frac{2}{\pi}\left(\int_0^{\to a}+\int_{\to a}^{\to\infty}\right)\cos xu\,du\int_0^{\to\infty}f(t)\cos ut\,dt.$$
Proceeding as in Theorem 8, we find that, in the first repeated integral, the integral over $0\leqslant u\leqslant\lambda$ $(\lambda>a)$ may be inverted if
$$\int_T^{\to\infty}\frac{g(t)\sin at}{t}\{\cos(a+\delta)x\sin(a+\delta)t-\cos(a-\delta)x\sin(a-\delta)t\}\,dt\to0$$

as $\delta \to 0$, i.e.

$$\cos ax \cos \delta x \int\limits_T^{\to\infty} \frac{g(t)}{t} \sin at \cos at \sin \delta t \, dt -$$
$$- \sin ax \sin \delta x \int\limits_T^{\to\infty} \frac{g(t)}{t} \sin^2 at \cos \delta t \, dt \to 0.$$

The first integral $\to 0$ by uniform convergence. In the second,

$$\int\limits_T^{\to\infty} \frac{g(t)}{t} \cos 2at \cos \delta t \, dt$$

is uniformly convergent, and so tends to a finite limit; and

$$\int\limits_T^{\to\infty} \frac{g(t)}{t} \cos \delta t \, dt = O\left(\int\limits_T^{1/\delta} \frac{dt}{t}\right) + O\left\{\delta g\left(\frac{1}{\delta}\right) \int\limits_{1/\delta}^{\xi} \cos \delta t \, dt\right\}$$
$$= O(\log 1/\delta) + O\{g(1/\delta)\} = O(\log 1/\delta),$$

and $\sin \delta x \log 1/\delta \to 0$. This proves the first part.

In the second part of the theorem we have to consider

$$\int\limits_T^{\to\infty} \frac{g(t) \sin at}{t} \{\cos ax \sin at - \cos(a-\delta)x \sin(a-\delta)t\} \, dt,$$

and this involves uniformly convergent terms, as before, and terms involving

$$\int\limits_T^{\infty} \frac{g(t)}{t} \, dt.$$

This proves the second part.

There is also a similar pair of theorems in which sines and cosines are interchanged.

Theorem 10.† *Let* $f(t) = g(t)h(t)$, *where* $g(t)$ *is ultimately steadily decreasing to zero,* $g(t)/(1+|t|)$ *belongs to* $L(-\infty,\infty)$, *and* $h(t)$ *is periodic (with period* a*) and integrable over a period. Let* $f(t)$ *be of bounded variation, or satisfy one of the alternative conditions in the neighbourhood of the point* $t = x$. *Then Fourier's formula holds in the sense that*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \sum_{n=0}^{\infty} \frac{1}{\pi} \int\limits_{\to 2n\pi/a}^{\to(2n+2)\pi/a} du \int\limits_{\to-\infty}^{\to\infty} f(t) \cos u(x-t) \, dt.$$

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† Hahn (2).

If $g(t)$ is steadily decreasing,

$$\int_{na}^{(n+1)a} \frac{|f(t)|}{1+t}\,dt = \int_{na}^{(n+1)a} \frac{|g(t)h(t)|}{1+t}\,dt$$

$$\leqslant \frac{g(na)}{1+na} \int_{na}^{(n+1)a} |h(t)|\,dt < K\frac{g(na)}{1+na} < K \int_{(n-1)a}^{na} \frac{g(t)}{1+t}\,dt.$$

Hence $f(t)/(1+|t|)$ belongs to $L(-\infty,\infty)$. Hence, as in Theorem 5,

$$\frac{1}{2x}\{f(x+0)+f(x-0)\} = \frac{1}{\pi} \int_{0}^{\to\infty} \{-b_1(u)\cos xu+a_1(u)\sin xu\}\,du,$$

where $a_1(u)$, $b_1(u)$ are defined by (1.9.4), (1.9.5).

Also

$$\sum_{\nu=m}^{n} \int_{\nu a}^{(\nu+1)a} h(x)e^{ix\nu}\,dx = \sum_{\nu=m}^{n} \int_{0}^{a} h(x)e^{i(\nu a+x)y}\,dx$$

$$= \frac{e^{imay}-e^{i(n+1)ay}}{1-e^{iay}} \int_{0}^{a} h(x)e^{ixy}\,dx,$$

which is bounded in any interval not containing one of the points $y = 0, \pm\frac{2\pi}{a}, \pm\frac{4\pi}{a},\dots$. Hence the integrals

$$\int_{x_1}^{x_2} h(x)\,{\cos \atop \sin}\,xy\,dx$$

are bounded, for all x_1 and x_2, in any such interval of values of y.

It then follows from the second mean-value theorem that the integrals

$$\frac{1}{\pi} \int_{\to-\infty}^{\to\infty} g(x)h(x)\,{\cos \atop \sin}\,xy\,dx$$

are uniformly convergent in any such interval, to $a(y)$, $b(y)$, say; $a_1(y)$, $b_1(y)$ are the integrals of $a(y)$, $b(y)$ in the interior of such an interval; and

$$\int_{2n\pi/a}^{(2n+2)\pi/a} b_1(u)\cos xu\,du$$

$$= \left[b_1(u)\frac{\sin xu}{x}\right]_{2n\pi/a}^{(2n+2)\pi/a} - \frac{1}{x} \int_{\to 2n\pi/a}^{\to(2n+2)\pi/a} b(u)\sin xu\,du,$$

and similarly for $a(u)$.

Also a_1 and b_1 are continuous, so that, on summation, all the integrated terms cancel, and the result follows.

1.12. Oscillating functions. In the above theorems our conditions on $f(x)$ are mostly restrictions on its oscillations. We shall next obtain a case of Fourier's theorem which depends on the oscillations of $f(x)$ being sufficiently rapid, provided they are of a regular kind.

THEOREM 11.[†] *Let* $f(t) = \phi(t)\cos\psi(t)$ *or* $\phi(t)\sin\psi(t)$, *where* $\phi(t)$ *is integrable over any finite interval, continuous and of bounded variation in any interval not containing the origin, and ultimately monotonic. Let* $\psi(t)$ *be twice differentiable,* $\psi'(t)$ *and* $\psi'(t)/\phi(t)$ *ultimately increasing steadily to infinity, and*

$$\phi(t) = o\{t\sqrt{\psi''(t)}\}. \tag{1.12.1}$$

Then

$$f(x) = \frac{2}{\pi}\int\limits_0^{\to\infty} \genfrac{}{}{0pt}{}{\cos}{\sin} xu\, du \int\limits_0^{\to\infty} f(t) \genfrac{}{}{0pt}{}{\cos}{\sin} ut\, dt \tag{1.12.2}$$

if (i) $\psi''(t)$ *is non-decreasing,* $\psi''(t+1) = O\{\psi''(t)\}$, $\phi(t+1) = O\{\phi(t)\}$, *or* (ii) $\psi''(t)$ *is decreasing,* $t\psi''(t) > K$, $\phi(2t) = O\{\phi(t)\}$.

We use the following lemma.

LEMMA. *If* $k(t)/h'(t)$ *is monotonic, and* $g(t)$ *steadily decreasing, then*

$$\int\limits_a^b k(t)g(t) \genfrac{}{}{0pt}{}{\cos}{\sin} h(t)\, dt = O\left\{g(a)\max\frac{k(t)}{h'(t)}\right\}.$$

Using the second mean-value theorem repeatedly, we have, if $k(t)/h'(t)$ increases,

$$\begin{aligned}
\int\limits_a^b k(t)g(t)\cos h(t)\, dt &= \int\limits_a^b \frac{k(t)}{h'(t)} g(t)h'(t)\cos h(t)\, dt \\
&= \frac{k(b)}{h'(b)} \int\limits_\alpha^b g(t)h'(t)\cos h(t)\, dt \quad (a < \alpha < b) \\
&= \frac{k(b)}{h'(b)} g(\alpha) \int\limits_\alpha^\beta h'(t)\cos h(t)\, dt \quad (\alpha < \beta < b) \\
&= \frac{k(b)}{h'(b)} g(\alpha)\{\sin h(\beta) - \sin h(\alpha)\},
\end{aligned}$$

and similarly for the other cases. Hence the result.

[†] Suggested by Landau, *Vorlesungen über Zahlentheorie*, Satz 413.

The inner integral in (1.12.2) is convergent if

$$\lim_{T,T'\to\infty}\int_T^{T'}\phi(t)\,\frac{\cos}{\sin}\{\psi(t)\pm ut\}\,dt=0. \tag{1.12.3}$$

Now
$$\frac{\phi(t)}{\psi'(t)\pm u}=\frac{\phi(t)}{\psi'(t)}\,\frac{\psi'(t)}{\psi'(t)\pm u},$$
and the first factor tends steadily to 0, while the second factor is
$$1\mp\frac{u}{\psi'(t)\pm u},$$
and the last term is steadily decreasing in absolute value. Hence (1.12.3) follows from the lemma; and the convergence is plainly uniform over any finite range of values of u. Hence

$$\int_0^{\lambda}\cos xu\,du\int_0^{\to\infty}f(t)\cos ut\,dt=\int_0^{\to\infty}f(t)\,dt\int_0^{\lambda}\cos xu\cos ut\,du.$$

As in previous cases, it is now sufficient to prove that
$$\lim_{\lambda\to\infty}\int_T^{\infty}f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt=0$$
for a sufficiently large T.

Take Case (i), and suppose that $\phi(t)$ is non-decreasing for $t\geqslant T$, and consider, e.g.

$$\int_T^{\infty}\frac{\phi(t)}{t-x}\cos\{\psi(t)+\lambda(x-t)\}\,dt=\int_T^{\frac{1}{2}t_0}+\int_{\frac{1}{2}t_0}^{t_0-\delta}+\int_{t_0-\delta}^{t_0+\delta}+\int_{t_0+\delta}^{\infty}$$
$$=J_1+J_2+J_3+J_4,$$

where $\psi'(t_0)=\lambda$. Now $\phi(t)/\{\lambda-\psi'(t)\}$ is steadily increasing for $t<t_0$; hence, by the lemma,

$$J_1=O\left\{\frac{\phi(\frac{1}{2}t_0)}{\psi'(t_0)-\psi'(\frac{1}{2}t_0)}\right\}=O\left\{\frac{\phi(\frac{1}{2}t_0)}{t_0\psi''(\frac{1}{2}t_0)}\right\}=o(1),$$

$$J_2=O\left\{\frac{\phi(t_0-\delta)}{t_0\{\psi'(t_0)-\psi'(t_0-\delta)\}}\right\}=O\left\{\frac{\phi(t_0)}{t_0\,\delta\psi''(t_0-\delta)}\right\}=O\left\{\frac{\phi(t_0)}{t_0\,\delta\psi''(t_0)}\right\} \tag{1.12.4}$$

provided that $\delta=O(1)$. For $t>t_0$
$$\frac{\phi(t)}{\psi'(t)-\lambda}=\frac{\phi(t)}{\psi'(t)}\left\{1+\frac{\lambda}{\psi'(t)-\lambda}\right\}$$
is decreasing, and J_4 also satisfies (1.12.4).

Lastly, $$J_3 = O\left\{\int\limits_{t_0-\delta}^{t_0+\delta} \left|\frac{\phi(t)}{x-t}\right| dt\right\} = O\left\{\frac{\delta\phi(t_0)}{t_0}\right\}.$$

Taking $\delta = \{\psi''(t_0)\}^{-\frac{1}{2}}$, the required result follows from (1.12.1). The corresponding integral with $-\lambda$ instead of λ is simpler, there being now no need to introduce t_0. The result therefore follows in this case.

If $\phi(t)$ is decreasing, we obtain instead of (1.12.4)

$$J_2 = O\left\{\frac{\phi(T)}{t_0\,\delta\psi''(t_0)}\right\},$$

and the result follows with $\delta = 1$.

The argument in Case (ii) is substantially the same. Examples are $\phi(t) = e^{\frac{1}{2}t}$, $\psi(t) = e^t$; $\phi(t) = 1$, $\psi(t) = t\log t$.

1.13. The constant in Fourier's formula. The constant π enters into Fourier's formula, according to our proof, through the formula

$$\int\limits_0^{\to\infty} \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi.$$

If we take the value of this integral as our fundamental constant, and denote it by C, Fourier's cosine formula, for example, is

$$f(x) = \frac{1}{C}\int\limits_0^{\to\infty} \cos xu\,du \int\limits_0^{\infty} \cos ut\,f(t)\,dt.$$

The values of other familiar integrals are then obtained in terms of C; for example, taking $f(t) = e^{-t}$, and $x = 0$, we obtain

$$1 = \frac{1}{C}\int\limits_0^{\to\infty} du \int\limits_0^{\infty} \cos ut\,e^{-t}\,dt = \frac{1}{C}\int\limits_0^{\infty} \frac{du}{1+u^2},$$

so that $$\int\limits_0^{\infty} \frac{du}{1+u^2} = C.$$

Taking $f(x) = x^{-\frac{1}{2}}$ $(x > 0)$, we obtain (by Theorem 6)

$$\frac{1}{\sqrt{x}} = \frac{1}{C}\int\limits_0^{\to\infty} \cos xu\,du \int\limits_0^{\to\infty} \frac{\cos ut}{\sqrt{t}}\,dt$$

$$= \frac{1}{C}\int\limits_0^{\to\infty} \frac{\cos xu}{\sqrt{u}}\,du \int\limits_0^{\to\infty} \frac{\cos y}{\sqrt{y}}\,dy = \frac{1}{C\sqrt{x}}\left(\int\limits_0^{\to\infty} \frac{\cos y}{\sqrt{y}}\,dy\right)^2,$$

so that
$$\int\limits_0^{\to\infty} \frac{\cos y}{\sqrt{y}}\,dy = \sqrt{C}.$$

Many other such examples may be derived from the formulae of Chapter VII.

Later, § 1.27, we shall give another proof of a case of Fourier's formula, in which the constant π comes from the theorem of residues.

1.14. Fourier's single-integral formula. This is the formula (1.1.7). Conditions for its validity are suggested by several of the foregoing theorems; but it holds still more generally, since now it is not necessary for the Fourier transform of $f(x)$ to exist.

THEOREM 12.† *The formula*
$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \lim_{\lambda\to\infty}\frac{1}{\pi}\int\limits_{\to-\infty}^{\to\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt$$
holds if

i (a) $f(x)/(1+|x|)$ *belongs to* $L(-\infty,\infty)$,

or i (b) $f(x)/x$ *is of bounded variation in* (a,∞) *and* $(-\infty,-a)$ *for some positive* a, *and tends to* 0 *at infinity,*

or i (c) $\displaystyle\frac{1}{x}\int\limits_1^x f(t)\,dt$ *is of bounded variation in* (a,∞) *and tends to* 0 *at infinity, and a similar condition holds in* $(-\infty,-a)$;

and (ii) *in an interval including* x, $f(t)$ *is of bounded variation, or satisfies one of the other conditions for the validity of Fourier's series or integral.*

After the analysis of § 1.9 it is sufficient to prove that we can choose T so large that
$$\left|\int\limits_T^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt\right| < \epsilon$$
for all values of $\lambda > \lambda_0$, with a similar condition for $(-\infty,-T)$. This is clearly true if i (a) holds. It follows from the second mean-value theorem as in § 1.10 if i (b) holds.

To prove that i (c) is sufficient, let
$$\phi(x) = \frac{1}{x}\int\limits_1^x f(t)\,dt.$$

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† Prasad (1), Pringsheim (1), Hobson (1).

Then $$f(x) = x\phi'(x)+\phi(x),$$
and $x\phi'(x)$ satisfies i (a), while $\phi(x)$ satisfies i (b). Hence the result.

Condition i (c) includes i (a); for
$$\phi'(x) = \frac{f(x)}{x}-\frac{1}{x^2}\int\limits_1^x f(t)\,dt.$$
The first term belongs to $L(1,\infty)$ if i (a) holds; and so does the second, since
$$\int\limits_1^\xi \frac{dx}{x^2}\int\limits_1^x |f(t)|\,dt = -\frac{1}{\xi}\int\limits_1^\xi |f(t)|\,dt+\int\limits_1^\xi \frac{1}{x}|f(x)|\,dx$$
$$\leqslant \int\limits_1^\xi \frac{1}{x}|f(x)|\,dx < K$$
as $\xi \to \infty$. Hence $\phi'(x)$ belongs to $L(1,\infty)$, and hence i (c) holds. On the other hand, i (c) does not include i (b).

1.15. Summability of integrals. We say that the integral $\int\limits_0^\infty f(x)\,dx$ is summable (C,α), where $\alpha \geqslant 0$, to the sum I, if
$$\lim_{\lambda\to\infty}\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)^\alpha f(x)\,dx = I.$$
The case $\alpha = 0$ is ordinary convergence. In the case $\alpha = 1$ we have
$$\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)f(x)\,dx = \frac{1}{\lambda}\int\limits_0^\lambda dx\int\limits_0^x f(y)\,dy,$$
a form analogous to the sum
$$\frac{s_1+s_2+\dots+s_n}{n}$$
in the definition of $(C, 1)$ summability of a series. The whole process is analogous to the C-summation of series, which is too well known to need much discussion here. The main points are (i) the process is more general than ordinary convergence; for example,
$$\lim_{\lambda\to\infty}\int\limits_0^\lambda \sin ax\,dx \quad (a > 0)$$

does not exist, but

$$\lim_{\lambda\to\infty}\int_0^{\lambda}\left(1-\frac{x}{\lambda}\right)\sin ax\,dx = \lim_{\lambda\to\infty}\left(\frac{1}{a}-\frac{\sin a\lambda}{a^2\lambda}\right) = \frac{1}{a};$$

and (ii) that it is consistent with ordinary convergence, in the sense that, if an integral is convergent, it is summable (C,α) to the same value for every $\alpha>0$. This is a particular case of the following theorem.

If an integral is summable (C,α), where $\alpha\geqslant 0$, it is summable (C,β), where $\beta>\alpha$, to the same value.

Let
$$\phi(\lambda,\alpha)=\int_0^{\lambda}\left(1-\frac{x}{\lambda}\right)^{\alpha}f(x)\,dx.$$

Then if $\beta>\alpha$,

$$\int_0^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\phi(\lambda,\alpha)\,d\lambda = \int_0^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\,d\lambda\int_0^{\lambda}\left(1-\frac{x}{\lambda}\right)^{\alpha}f(x)\,dx$$

$$= \int_0^{\mu}f(x)\,dx\int_x^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\left(1-\frac{x}{\lambda}\right)^{\alpha}\lambda^{\alpha}\,d\lambda$$

$$= \frac{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}{\Gamma(\beta+1)}\mu^{\alpha+1}\int_0^{\mu}f(x)\left(1-\frac{x}{\mu}\right)^{\beta}dx,$$

i.e.
$$\phi(\mu,\beta)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\frac{1}{\mu^{\alpha+1}}\int_0^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\phi(\lambda,\alpha)\,d\lambda.$$

Hence
$$\phi(\mu,\beta)-I=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\frac{1}{\mu^{\alpha+1}}\int_0^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\{\phi(\lambda,\alpha)-I\}\,d\lambda.$$

If $\phi(\lambda,\alpha)\to I$, suppose that $|\phi(\lambda,\alpha)-I|\leqslant M$ for all λ, and $\leqslant\epsilon$ for $\lambda\geqslant\Delta$. Then

$$|\phi(\mu,\beta)-I|\leqslant\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\left\{\frac{\epsilon}{\mu^{\alpha+1}}\int_{\Delta}^{\mu}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\,d\lambda+\right.$$

$$\left.+\frac{M}{\mu^{\alpha+1}}\int_0^{\Delta}\left(1-\frac{\lambda}{\mu}\right)^{\beta-\alpha-1}\lambda^{\alpha}\,d\lambda\right\}.$$

The first term is $\leqslant \epsilon$ (it is ϵ if $\Delta = 0$), and the second is $O(\mu^{-\alpha-1})$ for a fixed Δ. It follows by choosing first Δ and then μ that $\phi(\mu,\beta) \to I$, the required result.

1.16. Summability of Fourier integrals. We have formally

$$\frac{1}{\pi}\int_0^\lambda \left(1-\frac{u}{\lambda}\right) du \int_{-\infty}^{\infty} f(t)\cos u(x-t)\, dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_0^\lambda \left(1-\frac{u}{\lambda}\right)\cos u(x-t)\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{2\sin^2\frac{1}{2}\lambda(x-t)}{\lambda(x-t)^2}\, dt. \tag{1.16.1}$$

This integral is analogous to Fejér's integral in the theory of Fourier series. We shall deal with it as a particular case of the following theorem.

Theorem 13. *Let*

$$K(x,y,\delta) = O\left(\frac{1}{\delta}\right) \qquad (|x-y| \leqslant \delta) \tag{1.16.2}$$

$$= O\left(\frac{\delta^\alpha}{|x-y|^{\alpha+1}}\right) \qquad (|x-y| > \delta) \tag{1.16.3}$$

for some positive α, *and let*

$$\lim_{\delta\to 0}\int_x^\infty K(x,y,\delta)\, dy = \tfrac{1}{2}, \qquad \lim_{\delta\to 0}\int_{-\infty}^x K(x,y,\delta)\, dy = \tfrac{1}{2}.$$

Let $f(x)/(1+|x|^{\alpha+1})$ *belong to* $L(-\infty,\infty)$. *Then*

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} K(x,y,\delta)f(y)\, dy = \tfrac{1}{2}\{\phi(x)+\psi(x)\} \tag{1.16.4}$$

wherever

$$\int_0^h |f(x+t)-\phi(x)|\, dt = o(h), \qquad \int_0^h |f(x-t)-\psi(x)|\, dt = o(h) \tag{1.16.5}$$

as $h \to +0$. *The result therefore holds* (i) *with* $\phi(x) = f(x+0)$, $\psi(x) = f(x-0)$ *wherever these expressions have a meaning*, (ii) *with* $\phi(x) = \psi(x) = f(x)$ *wherever* $f(x)$ *is continuous, and* (iii) *with*

$$\phi(x) = \psi(x) = f(x)$$

for almost all values of x.

It is sufficient to prove that

$$\lim_{\delta\to 0}\int_x^\infty K(x,y,\delta)\{f(y)-\phi(x)\}\,dy = 0,$$

together with a similar result with $\psi(x)$; and by (1.16.2) and (1.16.3) this is true if

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_x^{x+\delta}|f(y)-\phi(x)|\,dy = 0$$

and

$$\lim_{\delta\to 0}\delta^\alpha\int_{x+\delta}^\infty\frac{|f(y)-\phi(x)|}{|x-y|^{\alpha+1}}\,dy = 0.$$

The first part follows at once from (1.16.5). Next, let

$$\chi(h) = \int_0^h|f(x+t)-\phi(x)|\,dt \leqslant \epsilon h$$

for $h \leqslant \eta$. Then

$$\delta^\alpha\int_{x+\delta}^{x+\eta}\frac{|f(y)-\phi(x)|}{|x-y|^{\alpha+1}}\,dy = \delta^\alpha\int_\delta^\eta\frac{|f(x+t)-\phi(x)|}{t^{\alpha+1}}\,dt$$

$$= \delta^\alpha\left[\frac{\chi(t)}{t^{\alpha+1}}\right]_\delta^\eta+(\alpha+1)\delta^\alpha\int_\delta^\eta\frac{\chi(t)}{t^{\alpha+2}}\,dt$$

$$\leqslant \epsilon+\epsilon(\alpha+1)\delta^\alpha\int_\delta^\eta\frac{dt}{t^{\alpha+1}} < \epsilon\left(1+\frac{\alpha+1}{\alpha}\right).$$

Having fixed η, plainly

$$\lim_{\delta\to 0}\delta^\alpha\int_{x+\eta}^\infty\frac{|f(y)-\phi(x)|}{|x-y|^{\alpha+1}}\,dy = 0.$$

This proves the theorem.

As a particular case, let $\delta = 1/\lambda$, and

$$K(x,y,\delta) = \frac{2\sin^2\frac{1}{2}\lambda(x-y)}{\pi\lambda(x-y)^2}.$$

The conditions of the above theorem are satisfied (with $\alpha = 1$). We therefore deduce† the analogue of Fejér's theorem on Fourier series:

THEOREM 14. *Let $f(t)$ belong to $L(-\infty,\infty)$. Then the integral*

$$\frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty f(t)\cos u(x-t)\,dt$$

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† Hardy (5).

is summable $(C,1)$ *to* $\frac{1}{2}\{f(x+0)+f(x-0)\}$ *wherever this expression has a meaning*; *to* $f(x)$ *wherever* $f(x)$ *is continuous*; *and to* $f(x)$ *for almost all values of* x.

An obvious corollary is that *if* $f(t)$ *is* $L(-\infty,\infty)$, *and* $a(u)$, $b(u)$, *defined by* (1.1.3), *are* 0 *for all* u, *then* $f(x)=0$ *for almost all* x.

As another particular case, let

$$K\left(x,y,\frac{1}{\lambda}\right)=\frac{1}{\pi}\int\limits_0^{\lambda}\left(1-\frac{u}{\lambda}\right)^{\alpha}\cos u(x-y)\,du$$

$$=\frac{\lambda}{\pi}\int\limits_0^{1}(1-v)^{\alpha}\cos\lambda v(x-y)\,dv$$

$$=\frac{\alpha}{\pi(x-y)}\int\limits_0^{1}(1-v)^{\alpha-1}\sin\lambda v(x-y)\,dv$$

$$=\frac{\alpha}{\pi\lambda^{\alpha}|x-y|^{\alpha+1}}\int\limits_0^{\lambda|x-y|}\frac{\sin\{\lambda|x-y|-w\}}{w^{1-\alpha}}\,dw.$$

The second formula shows that $K(x,y,1/\lambda)$ is $O(\lambda)$, and the fourth that it is $O(\lambda^{-\alpha}|x-y|^{-\alpha-1})$ for $0<\alpha<1$. Also

$$\int\limits_x^{\to\infty}K\left(x,y,\frac{1}{\lambda}\right)dy=\frac{\alpha}{\pi}\int\limits_x^{\to\infty}\frac{dy}{y-x}\int\limits_0^{1}(1-v)^{\alpha-1}\sin\lambda v(y-x)\,dv$$

$$=\frac{\alpha}{\pi}\int\limits_0^{1}(1-v)^{\alpha-1}\,dv\int\limits_0^{\to\infty}\frac{\sin\lambda vt}{t}\,dt=\tfrac{1}{2}\alpha\int\limits_0^{1}(1-v)^{\alpha-1}\,dv=\tfrac{1}{2}.$$

Hence

THEOREM 15. *Theorem* 14 *is still true if* $(C,1)$ *is replaced by* (C,α), *where* $0<\alpha<1$.

1.17. Cauchy's singular integral.[†] In the theorem of the previous section we have replaced Fourier's formula by a limit of the form

$$\lim_{\delta\to 0}\frac{1}{\pi}\int\limits_0^{\infty}\phi(\delta u)\,du\int\limits_{-\infty}^{\infty}f(t)\cos u(x-t)\,dt. \tag{1.17.1}$$

For summability (C,α) we take

$$\phi(u)=\begin{cases}(1-u)^{\alpha} & (0<u<1),\\ 0 & (u\geqslant 1).\end{cases} \tag{1.17.2}$$

[†] Cauchy (1), Sommerfeld (1), Hardy (4), (5).

Next let
$$\phi(u) = e^{-u}. \tag{1.17.3}$$
The integral in (1.17.1) is then formally
$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} e^{-\delta u}\cos u(x-t)\,du = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\delta}{\delta^2+(x-t)^2}\,dt. \tag{1.17.4}$$
Here
$$K(x,y,\delta) = \frac{1}{\pi}\frac{\delta}{\delta^2+(x-y)^2}, \tag{1.17.5}$$
and the conditions of Theorem 13 are again satisfied, with $\alpha = 1$. In particular it follows that (1.17.4) tends to $f(x)$ almost everywhere.

This result is the rigorous form of Cauchy's argument given in § 1.1, and the integral (1.17.4) may be called *Cauchy's singular integral.* The type of summability obtained is analogous to 'summability A' for series.

1.18. Weierstrass's singular integral.

Now let
$$\phi(u) = e^{-u^2}.$$
The integral (1.17.1) is then
$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} e^{-\delta^2u^2}\cos u(x-t)\,du = \frac{1}{2\delta\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)\exp\left\{-\frac{(x-t)^2}{4\delta^2}\right\}dt.$$
Here
$$K(x,y,\delta) = \frac{1}{2\delta\sqrt{\pi}}\exp\left\{-\frac{(x-y)^2}{4\delta^2}\right\}, \tag{1.18.1}$$
and the integral is known as *Weierstrass's singular integral.*†

The conditions of Theorem 13 are satisfied for any positive α; but in fact the result holds still more generally.

THEOREM 16. *If $K(x,y,\delta)$ is defined by* (1.18.1), *the results of Theorem* 13 *hold if $e^{-Cx^2}f(x)$ belongs to $L(-\infty,\infty)$ for some positive value of C (and so for all greater values).*

We argue as in § 1.16, with $\alpha = 1$ say, for the integrals over $(x, x+\delta)$ and $(x+\delta, x+\eta)$. It then remains to prove that, for fixed x and η,
$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{x+\eta}^{\infty} \exp\left\{-\frac{(x-y)^2}{4\delta^2}+Cy^2\right\}g(y)\,dy = 0,$$
where $g(y)$ is L. Now
$$-\frac{(x-y)^2}{4\delta^2}+Cy^2 = -\frac{(x-y)^2}{4\delta^2}+C(x-y)^2\frac{y^2}{(x-y)^2} \leqslant -\frac{(x-y)^2}{4\delta^2}+\frac{(x-y)^2}{8\delta^2}$$

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† Weierstrass (1), Hobson (1), Lebesgue (1), Hardy (6).

if
$$C \leqslant \frac{1}{8\delta^2}\frac{\eta^2}{(x+\eta)^2}.$$

Hence the modulus of the above expression does not exceed

$$\frac{1}{\delta}e^{-\eta^2/(8\delta^2)}\int\limits_{x+\eta}^{\infty}|g(y)|\,dy,$$

and this tends to 0.

1.19. General summability. If we merely require to deal with the case in which $f(x+0)$ and $f(x-0)$ exist, we can use the following simpler theorem.

THEOREM 17. *Let* $K(x,y,\delta) \geqslant 0$,

$$\lim_{\delta\to 0}\int\limits_{x}^{b} K(x,y,\delta)\,dy = \tfrac{1}{2}, \qquad \lim_{\delta\to 0}\int\limits_{a}^{x} K(x,y,\delta)\,dy = \tfrac{1}{2}; \tag{1.19.1}$$

and let
$$\lim_{\delta\to 0} K(x,y,\delta) = 0 \tag{1.19.2}$$

uniformly for all x *and* y *for which* $|x-y| \geqslant \epsilon > 0$, *and also, in the case* $a = -\infty$, $b = \infty$,

$$\lim_{\delta\to 0}\int\limits_{-\infty}^{x-\epsilon} K(x,y,\delta)\,dy = 0, \qquad \lim_{\delta\to 0}\int\limits_{x+\epsilon}^{\infty} K(x,y,\delta)\,dy = 0 \tag{1.19.3}$$

for any fixed positive ϵ.

Let $f(x)$ *belong to* $L(a,b)$. *Then*

$$\lim_{\delta\to 0}\int\limits_{a}^{b} K(x,y,\delta)f(y)\,dy = \tfrac{1}{2}\{f(x+0)+f(x-0)\} \tag{1.19.4}$$

wherever the right-hand side exists.

If f *is continuous at the point* x, (1.19.1) *can be replaced by*

$$\lim_{\delta\to 0}\int\limits_{-\infty}^{\infty} K(x,y,\delta)\,dy = 1. \tag{1.19.5}$$

We have to prove that

$$\int\limits_{x}^{b} K(x,y,\delta)\{f(y)-f(x+0)\}\,dy \to 0,$$

with a similar result for (a,x). This integral does not exceed in absolute value

$$\max_{x\leqslant y\leqslant x+\epsilon}|f(y)-f(x+0)|\int_x^b K(x,y,\delta)\,dy+$$

$$+\max_{y>x+\epsilon}K(x,y,\delta)\int_{x+\epsilon}^b |f(y)|\,dy+|f(x+0)|\int_{x+\epsilon}^b K(x,y,\delta)\,dy,$$

which tends to 0 by choosing first ϵ and then δ. Similarly for the other part.

The relevant parts of the summability theorems are clearly cases of this theorem. They may, however, be exhibited as direct consequences of the form of the summability factor; the general result is as follows.

THEOREM 18. *Let* $\phi(x)$ *belong to* $L(0,\infty)$ *and have only a finite number of maxima and minima in* $(0,\infty)$; *let* $\phi(+0)=1$; *and let* $\phi(x)$ *be the integral of* $\phi'(x)$, *which is ultimately negative non-decreasing. Let* $f(x)$ *belong to* $L(-\infty,\infty)$. *Then*

$$\lim_{\delta\to 0}\frac{1}{\pi}\int_0^\infty \phi(\delta u)\,du\int_{-\infty}^\infty f(t)\cos u(x-t)\,dt=\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

wherever the right-hand side exists.

This follows from the previous theorem if

$$K(x,y,\delta)=\frac{1}{\pi}\int_0^\infty \phi(\delta u)\cos u(x-y)\,du$$

has the properties stated.

Suppose first that $\phi'(x)$ is negative non-decreasing for all x. Then

$$K(x,y,\delta)=\frac{\delta}{\pi(x-y)}\int_0^\infty \{-\phi'(\delta u)\}\sin u(x-y)\,du$$

$$=\frac{\delta}{\pi|x-y|}\sum_{n=0}^\infty\int_{\frac{n\pi}{|x-y|}}^{\frac{(n+1)\pi}{|x-y|}}\{-\phi'(\delta u)\}\sin u|x-y|\,du.$$

The sum is positive, and its value does not exceed that of the first term. Hence

$$0\leqslant K(x,y,\delta)\leqslant\frac{\delta}{\pi|x-y|}\int_0^{\pi/|x-y|}\{-\phi'(\delta u)\}\,du$$

$$=\frac{1}{\pi|x-y|}\left\{\phi(+0)-\phi\left(\frac{\delta\pi}{|x-y|}\right)\right\},$$

which tends to 0 as $\delta \to 0$, uniformly for $|x-y| \geqslant \epsilon$. Hence (1.19.2) holds. Also

$$\int_x^\infty K(x,y,\delta)\,dy = -\frac{\delta}{\pi}\int_x^\infty \frac{dy}{x-y}\int_0^\infty \phi'(\delta u)\sin u(x-y)\,du$$
$$= -\frac{\delta}{\pi}\int_0^\infty \phi'(\delta u)\,du \int_x^\infty \frac{\sin u(x-y)}{x-y}\,dy$$
$$= -\tfrac{1}{2}\delta\int_0^\infty \phi'(\delta u)\,du = \tfrac{1}{2}\phi(+0) = \tfrac{1}{2},$$

the inversion being justified by dominated convergence, since $\phi'(\delta u)$ belongs to $L(0,\infty)$. Similarly for $(-\infty, x)$, and (1.19.1) follows.

Similarly,

$$\int_Y^\infty K(x,y,\delta)\,dy = -\frac{\delta}{\pi}\int_0^\infty \phi'(\delta u)\,du \int_Y^\infty \frac{\sin u(x-y)}{x-y}\,dy$$
$$= O\left(\delta\int_0^1 \{-\phi'(\delta u)\}\,du\right)+O\left(\frac{\delta}{Y}\int_1^\infty \{-\phi'(\delta u)\}\,du\right)$$
$$= O\{\phi(+0)-\phi(\delta)\}+O(Y^{-1}),$$

and (1.19.3) follows from (1.19.2) on choosing first Y sufficiently large, and then δ sufficiently small.

In the more general case, we can write $\phi(x) = \phi_1(x)-\phi_2(x)$, where ϕ_1' and ϕ_2' are negative non-decreasing. Let K_1, K_2 be the corresponding K-functions. Then K_1 and K_2 are positive, and satisfy (1.19.2); the integrals

$$\int_{-\infty}^\infty K_1(x,y,\delta)\,dy, \qquad \int_{-\infty}^\infty K_2(x,y,\delta)\,dy$$

are bounded; and (1.19.1) holds as before. These conditions are clearly sufficient for the theorem.

1.20. In all the particular cases considered, we have

$$K(x,y,\delta) = K(x-y,\delta),$$

where $K(u,\delta)$ is an even function of u,

$$\int_0^\infty K(u,\delta)\,du = \tfrac{1}{2},$$

and

$$\lim_{\delta\to 0}\int_\eta^\infty K(u,\delta)\,du = 0$$

for every positive η. With these conditions, not only does

$$\chi(x,\delta) = \int_{-\infty}^{\infty} K(x-y,\delta)f(y)\,dy$$

tend to $f(x)$ at particular points, but in the sense of the following theorem it tends to $f(x)$ *on the average.*

Theorem 19. *If $K(u,\delta)$ is positive and satisfies the above conditions and $f(x)$ is $L(-\infty,\infty)$, then*

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} |\chi(x,\delta)-f(x)|\,dx = 0.$$

For

$$\chi(x,\delta)-f(x) = \int_{-\infty}^{\infty} K(u,\delta)\{f(x-u)-f(x)\}\,du,$$

$$\int_{-\infty}^{\infty} |\chi(x,\delta)-f(x)|\,dx \leqslant \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} K(u,\delta)|f(x-u)-f(x)|\,du$$

$$= \int_{-\infty}^{\infty} K(u,\delta)\,du \int_{-\infty}^{\infty} |f(x-u)-f(x)|\,dx.$$

Now

$$\psi(u) = \int_{-\infty}^{\infty} |f(x-u)-f(x)|\,dx$$

is bounded for all u, and tends to 0 with u. Let $|\psi(u)| \leqslant \epsilon$ for $|u| \leqslant \eta$. Then

$$\left|\int_{-\eta}^{\eta} K(u,\delta)\psi(u)\,du\right| \leqslant \epsilon \int_{-\infty}^{\infty} K(u,\delta)\,du = \epsilon,$$

and, if $|\psi(u)| \leqslant M$,

$$\left|\int_{\eta}^{\infty} K(u,\delta)\psi(u)\,du\right| \leqslant M \int_{\eta}^{\infty} K(u,\delta)\,du,$$

which by hypothesis tends to 0 with δ. Similarly for $(-\infty,-\eta)$. Hence the result.

For example, in the case of $(C,1)$ summability,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{1-\cos\lambda(x-y)}{\lambda(x-y)^2}\,dy$$

converges to $f(x)$ on the average, in the above sense, as $\lambda \to \infty$. (Take $\lambda = 1/\delta$.)

1.21. Further summability theorems. In all the foregoing theorems we have imposed on $f(t)$ conditions which ensure the existence of the inner integral in Fourier's formula in some sense or other. We shall next prove a theorem in which, without imposing any particular condition, we merely assume that the inner integral exists.

Here we are not particularly concerned with the behaviour of $f(t)$ in finite intervals, and for the sake of simplicity we shall suppose that it is continuous.

THEOREM 20.[†] *Let $f(t)$ be integrable over any finite interval, continuous at $t = x$, and let*

$$\int_{\to-\infty}^{\to\infty} f(t)\cos u(x-t)\,dt$$

converge uniformly over any finite interval of values of u. If the limit is $g(x,u)$, then

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\lambda \left(1-\frac{u}{\lambda}\right)g(x,u)\,du = f(x).$$

We have

$$\int_0^\lambda \left(1-\frac{u}{\lambda}\right)g(x,u)\,du = \int_{\to-\infty}^{\to\infty} f(t)\,dt\int_0^\lambda \left(1-\frac{u}{\lambda}\right)\cos u(x-t)\,du$$

$$= \int_{\to-\infty}^{\to\infty} f(t)\frac{2\sin^2\frac{1}{2}\lambda(x-t)}{\lambda(x-t)^2}\,dt = \int_{\to-\infty}^{-T} + \int_{-T}^{T} + \int_{T}^{\to\infty} = J_1+J_2+J_3,$$

say, where $T > |x|$. The inversion is justified by uniform convergence.

Let $$f_1(t) = \int_0^t f(v)\,dv.$$

By the case $u = 0$ of the data, $f_1(t)$ is bounded, say $|f_1(t)| \leqslant M$. Now

$$J_3 = -f_1(T)\frac{2\sin^2\frac{1}{2}\lambda(x-T)}{\lambda(x-T)^2} - \int_T^\infty f_1(t)\frac{\sin\lambda(x-t)}{(x-t)^2}\,dt - \int_T^\infty f_1(t)\frac{4\sin^2\frac{1}{2}\lambda(x-t)}{\lambda(x-t)^3}\,dt$$

[†] The analogue for Mellin integrals was proved by Hardy (8). See Macphail and Titchmarsh (1).

on integrating by parts. Hence

$$|J_3| \leqslant \frac{2M}{\lambda(T-x)^2}+M\int_T^\infty \frac{dt}{(t-x)^2}+M\int_T^\infty \frac{dt}{\lambda(t-x)^3} = \frac{5M}{2\lambda(T-x)^2}+\frac{M}{T-x}.$$

This can be made arbitrarily small by choice of T, for all $\lambda > \lambda_0$; and a similar argument applies to J_1. Having fixed T, $J_2 \to \pi f(x)$ as in § 1.16. This proves the theorem.

1.22. The general result of the above type appears to be that, if the inner integral is summable (C, k), the outer integral is summable $(C, k+1)$. The above theorem is the case $k = 0$, and we shall next prove the case $k = 1$. The proof of the general case does not seem to require any new idea, but it would be rather laborious to write out.

THEOREM 21. *Let $f(t)$ be integrable over any finite interval, continuous at $t = x$, and let*

$$\lim_{\mu\to\infty}\int_0^\mu \left(1-\frac{t}{\mu}\right)f(t)\cos u(x-t)\,dt$$

and

$$\lim_{\mu\to\infty}\int_{-\mu}^0 \left(1-\frac{|t|}{\mu}\right)f(t)\cos u(x-t)\,dt$$

exist, uniformly over any finite interval of values of u. If the sum of these limits is $g(x, u)$,

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^2 g(x,u)\,du = f(x).$$

It will be sufficient to consider the case where $f(t) = 0$ for $t < 0$. By uniform convergence

$$\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^2 g(x,u)\,du = \lim_{\mu\to\infty}\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^2 du\int_0^\mu \left(1-\frac{t}{\mu}\right)f(t)\cos u(x-t)\,dt.$$

The repeated integral is equal to

$$\int_0^\mu \left(1-\frac{t}{\mu}\right)f(t)\,dt\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^2\cos u(x-t)\,du$$

$$= \int_0^\mu \left(1-\frac{t}{\mu}\right)f(t)\left\{\frac{2}{\lambda(x-t)^2}-\frac{2\sin\lambda(x-t)}{\lambda^2(x-t)^3}\right\}dt = \int_0^T + \int_T^\mu = J_1+J_2,$$

say, where $|x| < T < \mu$. Define $f_1(t)$ as before, and let

$$f_2(t) = \int_0^t f_1(v)\,dv = t\int_0^t \left(1-\frac{v}{t}\right)f(v)\,dv.$$

Then $f_2(t)/t$ is bounded, as a particular case of the data.

On integrating by parts twice, we obtain

$$J_2 = -\left(1-\frac{T}{\mu}\right)f_1(T)\phi(T)+\frac{1}{\mu}\{f_2(\mu)\phi(\mu)-f_2(T)\phi(T)\}+$$

$$+\left(1-\frac{T}{\mu}\right)f_2(T)\phi'(T)-\frac{2}{\mu}\int_T^\mu f_2(t)\phi'(t)\,dt+\int_T^\mu\left(1-\frac{t}{\mu}\right)f_2(t)\phi''(t)\,dt,$$

where

$$\phi(t) = \frac{2}{\lambda(x-t)^2}-\frac{2\sin\lambda(x-t)}{\lambda^2(x-t)^3},$$

$$\phi'(t) = \frac{4}{\lambda(x-t)^3}+\frac{2\cos\lambda(x-t)}{\lambda(x-t)^3}-\frac{6\sin\lambda(x-t)}{\lambda^2(x-t)^4},$$

$$\phi''(t) = \frac{12}{\lambda(x-t)^4}+\frac{2\sin\lambda(x-t)}{(x-t)^3}+\frac{12\cos\lambda(x-t)}{\lambda(x-t)^4}-\frac{24\sin\lambda(x-t)}{\lambda^2(x-t)^5}.$$

Making $\mu \to \infty$, we obtain

$$J_1 \to \int_0^T f(t)\phi(t)\,dt,$$

$$J_2 \to -f_1(T)\phi(T)+f_2(T)\phi'(T)+\int_T^\infty f_2(t)\phi''(t)\,dt.$$

We can choose T so large that the last two terms are arbitrarily small for all $\lambda > \lambda_0$. Having fixed T, $f_1(T)\phi(T) \to 0$ as $\lambda \to \infty$, and

$$\int_0^T f(t)\phi(t)\,dt \to \pi f(x)$$

by the theory of Fejér's integral, § 1.16, and the consistency theorem for C-summability, § 1.15. This proves the theorem.

1.23. We have seen that the $(C, 1)$ of Theorem 14 can be replaced by (C, α), where α is arbitrarily small. This is not true of Theorem 20; in neither Theorem 20 nor Theorem 21 can the order of summability of the outer integral be reduced. We shall now prove this by means of examples.

Let
$$I_\alpha(\lambda, t) = \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \cos ut\,du.$$

As in § 1.16, if $0 < \alpha < 1$,

$$I_\alpha(\lambda, t) = \frac{\alpha}{\lambda^\alpha t^{\alpha+1}} \int_0^{\lambda t} v^{\alpha-1} \sin(\lambda t - v)\, dv = O\left(\frac{1}{\lambda^\alpha t^{\alpha+1}}\right)$$

as $\lambda \to \infty$, $t \to \infty$. Also

$$\frac{\partial I}{\partial t} = \frac{\alpha\lambda^{1-\alpha}}{t^{\alpha+1}} \int_0^{\lambda t} v^{\alpha-1}\cos(\lambda t - v)\, dv - \frac{\alpha(\alpha+1)}{\lambda^\alpha t^{\alpha+2}} \int_0^{\lambda t} v^{\alpha-1}\sin(\lambda t - v)\, dv$$

$$= \frac{\alpha\lambda^{1-\alpha}}{t^{\alpha+1}} \int_0^{\to\infty} v^{\alpha-1}\cos(\lambda t - v)\, dv + O\left(\frac{1}{t^2}\right) + O\left(\frac{1}{\lambda^\alpha t^{\alpha+2}}\right)$$

$$= \Gamma(\alpha+1)\lambda^{1-\alpha} t^{-1-\alpha} \cos(\lambda t - \tfrac{1}{2}\pi\alpha) + O(t^{-2}).$$

Suppose that we try to prove Theorem 20 with (C, α), where $0 < \alpha < 1$, instead of $(C, 1)$. We encounter a term

$$J_3 = \int_T^\infty f(t) I_\alpha(\lambda, x-t)\, dt$$

$$= -f_1(T) I_\alpha(\lambda, x-T) + \int_T^\infty f_1(t) \frac{\partial}{\partial t} I_\alpha(\lambda, x-t)\, dt.$$

Take T fixed $(> |x|)$. Then everything is bounded except possibly the term

$$\Gamma(\alpha+1)\lambda^{1-\alpha} \int_T^\infty f_1(t) t^{-1-\alpha} \cos(\lambda x - \lambda t - \tfrac{1}{2}\pi\alpha)\, dt.$$

Let $$f(t) = 2^\nu \nu^{-2} \sin 2^\nu t \quad (\nu\pi \leqslant t < (\nu+1)\pi),$$

for $\nu = 1, 2, \ldots$, and $f(t) = 0$ elsewhere. Then

$$f_1(t) = \nu^{-2}(1 - \cos 2^\nu t) \quad (\nu\pi \leqslant t < (\nu+1)\pi).$$

Clearly

$$\int_0^T f(t) \cos u(x-t)\, dt = f_1(T) \cos u(x-T) - u \int_0^T f_1(t) \sin u(x-t)\, dt \to \text{limit}$$

as $T \to \infty$, uniformly with respect to u, so that the conditions of Theorem 20 are satisfied.

Let $\lambda = 2^\rho$. Then

$$\int_\pi^\infty f_1(t) t^{-1-\alpha} \cos \lambda t\, dt = \sum_{\nu=1}^\infty \frac{1}{\nu^2} \int_{\nu\pi}^{(\nu+1)\pi} \frac{(1 - \cos 2^\nu t)\cos 2^\rho t}{t^{\alpha+1}}\, dt.$$

The term $\nu = \rho$ is

$$\frac{1}{\rho^2}\int\limits_{\rho\pi}^{(\rho+1)\pi}\frac{\cos 2^\rho t}{t^{\alpha+1}}\,dt-\frac{1}{2\rho^2}\int\limits_{\rho\pi}^{(\rho+1)\pi}\frac{1+\cos 2^{\rho+1}t}{t^{\alpha+1}}\,dt$$

$$< -\frac{1}{2\rho^2(\rho+1)^\alpha\pi^\alpha}+O\left(\frac{1}{2^\rho}\right).$$

The remaining terms are

$$\sum_{\nu\neq\rho}\frac{1}{\nu^2}\int\limits_{\nu\pi}^{(\nu+1)\pi}\frac{\cos 2^\rho t-\frac{1}{2}\cos(2^\nu-2^\rho)t-\frac{1}{2}\cos(2^\nu+2^\rho)t}{t^{\alpha+1}}\,dt$$

$$= \sum_{\nu\neq\rho}O\left(\frac{1}{\nu^{\alpha+3}|2^\rho-2^\nu|}\right) = O\left(\frac{1}{2^\rho}\right).$$

Similarly,
$$\int\limits_{\pi}^{\infty}\frac{f_1(t)\sin\lambda t}{t^{\alpha+1}}\,dt = O\left(\frac{1}{2^\rho}\right).$$

Hence
$$|J_3| > A|\cos(\lambda x-\tfrac{1}{2}\pi\alpha)|\lambda^{1-\alpha}(\log\lambda)^{-\alpha-2}+O(1),$$

for $\lambda = 2^\rho$. Finally, the sequence $\cos(2^\rho x-\frac{1}{2}\pi\alpha)$ does not tend to 0, since, if one term is small, the next is approximately $-\cos\frac{1}{2}\pi\alpha$. Hence J_3 is unbounded.

Also, by Theorem 15, J_2 tends to a limit. It follows that, in Theorem 20, $(C, 1)$ cannot be replaced by (C, α), if $\alpha < 1$.

If $1 < \alpha < 2$, we can write

$$I_\alpha(\lambda, t) = \frac{\alpha}{\lambda t^2}-\frac{\alpha(\alpha-1)}{\lambda^\alpha t^{\alpha+1}}\int\limits_0^{\lambda t}v^{\alpha-2}\cos(\lambda t-v)\,dv.$$

Hence $\partial^2 I/\partial t^2$ contains a term

$$-\frac{\alpha(\alpha-1)\lambda^{2-\alpha}}{t^{\alpha+1}}\int\limits_0^{\lambda t}v^{\alpha-2}\cos(\lambda t-v)\,dv$$

$$= \frac{\Gamma(\alpha+1)\lambda^{2-\alpha}}{t^{\alpha+1}}\sin(\lambda t-\tfrac{1}{2}\pi\alpha)+O\left(\frac{1}{t^3}\right),$$

and, in the argument with (C, α) analogous to that of § 1.22, we obtain the term

$$\Gamma(\alpha+1)\lambda^{2-\alpha}\int\limits_T^\infty f_2(t)t^{-\alpha-1}\sin(\lambda x-\lambda t-\tfrac{1}{2}\pi\alpha)\,dt.$$

Let
$$f(t) = 2^{2\nu}\nu^{-2}\cos 2^\nu t \quad (\nu\pi \leqslant t < (\nu+1)\pi)$$

for $\nu = 1, 2, \ldots$, and $f(t) = 0$ elsewhere. Then

$$f_2(t) = \nu^{-2}(1-\cos 2^{\nu} t) \quad (\nu\pi \leqslant t < (\nu+1)\pi).$$

Since

$$\int_0^{\mu} \left(1-\frac{t}{\mu}\right) f(t)\cos u(x-t)\,dt = f_2(\mu)\frac{\cos u(x-\mu)}{\mu} - \int_0^{\mu} f_2(t)\left\{\frac{2u}{\mu}\sin u(x-t)+\left(1-\frac{t}{\mu}\right)u^2\cos u(x-t)\right\}dt,$$

which clearly tends to a limit as $\mu \to \infty$, the conditions of Theorem 21 are fulfilled.

The proof that the selected term is unbounded now proceeds as before. The remaining terms are easily seen to be bounded, and the desired result follows.

1.24. The integrated form of Fourier's formula. It is well known that the result of formally integrating a Fourier series term-by-term is true, whether the original series is convergent or not. The corresponding theorem for integrals is as follows.

THEOREM 22. *If $f(x)$ belongs to $L(-\infty,\infty)$, then*

$$\int_0^{\xi} f(x)\,dx = \frac{1}{\pi}\int_0^{\to\infty}\frac{du}{u}\int_{-\infty}^{\infty} f(t)\{\sin u(\xi-t)+\sin ut\}\,dt, \tag{1.24.1}$$

$$\int_0^{\xi} f(x)\,dx = \frac{1}{2\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\frac{e^{-i\xi u}-1}{-iu}\,du\int_{-\infty}^{\infty} f(t)e^{iut}\,dt \tag{1.24.2}$$

for all values of ξ; and

$$\int_0^{\xi} f(x)\,dx = \frac{2}{\pi}\int_0^{\to\infty}\frac{\sin\xi u}{u}\,du\int_0^{\infty} f(t)\cos ut\,dt, \tag{1.24.3}$$

$$\int_0^{\xi} f(x)\,dx = \frac{2}{\pi}\int_0^{\to\infty}\frac{1-\cos\xi u}{u}\,du\int_0^{\infty} f(t)\sin ut\,dt \tag{1.24.4}$$

for $\xi \geqslant 0$.

The formulae correspond to (1.1.1), (1.1.6), (1.1.4), (1.1.5) respectively. Consider for example (1.24.2). We have

$$\int_{-\lambda}^{\lambda}\frac{e^{-i\xi u}-1}{-iu}\,du\int_{-\infty}^{\infty} f(t)e^{iut}\,dt = \int_{-\infty}^{\infty} f(t)\,dt\int_{-\lambda}^{\lambda}\frac{e^{iu(t-\xi)}-e^{iut}}{-iu}\,du$$

by uniform convergence. The inner integral is

$$2\int_0^\lambda \frac{\sin tu-\sin(t-\xi)u}{u}\,du,$$

which is bounded for all t and λ, and, as $\lambda\to\infty$, tends to 2π $(0<t<\xi)$, 0 $(t<0$ or $t>\xi)$. The result therefore follows by dominated convergence. The other formulae are easily deduced from this, or proved in a similar way.

1.25. The complex form of Fourier's integral. The theory of the complex form of Fourier's integral is substantially the same as that of the forms already considered. We shall state briefly the most important results.

We have so far supposed that all the functions are real. There is, however, no additional difficulty in dealing with complex functions of a real variable, and it is natural to apply the complex form of the theorem to these. The extension of all the definitions is immediate; a complex function $f(x)$ is integrable, of bounded variation, etc., if its real and imaginary parts separately have these properties.

THEOREM 23. *Let $f(t)$ belong to $L(-\infty,\infty)$, and let it be of bounded variation in the neighbourhood of $t=x$. Then*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=\frac{1}{2\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}e^{-ixu}\,du\int_{-\infty}^{\infty}f(t)e^{iut}\,dt. \quad (1.25.1)$$

If $f(t)$ satisfies the conditions of Theorem 4 in the neighbourhood of $t=x$, the left-hand side may be replaced by $f(x)$.

We may invert by uniform convergence, and obtain

$$\int_{-\lambda}^{\lambda}e^{-ixu}\,du\int_{-\infty}^{\infty}f(t)e^{iut}\,dt=\int_{-\infty}^{\infty}f(t)\,dt\int_{-\lambda}^{\lambda}e^{-iu(x-t)}\,du$$

$$=2\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt,$$

and the result follows as in the proof of Theorem 3.

As a particular case, suppose in addition that $f(z)$ is analytic for $y\geqslant 0$, and $f(z)\to 0$ as $|z|\to\infty$ uniformly for $0\leqslant\arg z\leqslant\pi$. Then by Jordan's lemma (Whittaker and Watson, *Modern Analysis*, § 6.222) $F(u)=0$ for $u>0$. The formulae reduce by a change of variable to those of Laplace, (1.4.1), (1.4.2).

If we use the functions $F_+(w)$ and $F_-(w)$, we obtain a theorem in which $f(x)$ is less restricted at infinity.

THEOREM 24. *Let $e^{-c|t|}f(t)$ belong to $L(-\infty,\infty)$ for some positive c, so that $F_+(w)$, $F_-(w)$, defined by* (1.3.1), (1.3.2), *exist for $v \geqslant c$, $v \leqslant -c$, respectively. Then, if $f(t)$ satisfies conditions corresponding to those of Theorems* 3 *or* 4 *in the neighbourhood of $t = x$,*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda} F_+(w)e^{-ixw}\,dw+$$

$$+\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ib-\lambda}^{ib+\lambda} F_-(w)e^{-ixw}\,dw,$$

where $a \geqslant c$, $b \leqslant -c$.

Let $g(x) = e^{-ax}f(x)$ $(x>0)$, 0 $(x<0)$. Then by the previous theorem

$$\tfrac{1}{2}\{g(x+0)+g(x-0)\} = \frac{1}{2\pi}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda} e^{-ixu}\,du\int\limits_{-\infty}^{\infty} g(t)e^{iut}\,dt$$

$$= \frac{1}{2\pi}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda} e^{-ixu}\,du\int\limits_{0}^{\infty} f(t)e^{i(u+ia)t}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda} e^{-ixu}F_+(u+ia)\,du,$$

or
$$\tfrac{1}{2}e^{ax}\{g(x+0)+g(x-0)\} = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda} F_+(w)e^{-ixw}\,dw.$$

Similarly, if $h(x) = e^{bx}f(x)$ $(x<0)$, 0 $(x>0)$, then

$$\tfrac{1}{2}e^{-bx}\{h(x+0)+h(x-0)\} = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ib-\lambda}^{ib+\lambda} F_-(w)e^{-ixw}\,dw.$$

The result stated follows on addition.

1.26. Perron's formula.†

The formula known as Perron's formula in the theory of Dirichlet series can be deduced from Theorem 24.

THEOREM 25. *Let* $$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$
be convergent for $\sigma > \sigma_0$, where the λ_n are real and steadily increasing to infinity, and let $$A(x) = \sum_{\lambda_n \leqslant x} a_n.$$

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† See Hardy and Riesz, *The General Theory of Dirichlet's Series*, 12–14.

Then if $k > 0$, $k > \sigma_0$,

$$\tfrac{1}{2}\{A(x+0)+A(x-0)\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{k-iT}^{k+iT}\frac{f(s)}{s}e^{sx}\,ds.$$

Let $$A_0 = 0,\quad A_n = a_1+a_2+\ldots+a_n.$$

If $\beta > \sigma_0$, $\sum\limits_{1}^{m} a_\nu e^{-\lambda_\nu \beta}$ is bounded for all m, and hence, if also $\beta > 0$,

$$A_n = \sum_{\nu=1}^{n} a_\nu e^{-\lambda_\nu \beta}.e^{\lambda_\nu \beta} = O(e^{\lambda_n \beta}).$$

Hence, if N is the greatest n for which $\lambda_n \leqslant x$,

$$A(x) = A_N = O(e^{\lambda_N \beta}) = O(e^{\beta x}).$$

Hence for $\sigma > \beta$,

$$f(s) = \sum_{n=1}^{\infty}(A_n-A_{n-1})e^{-\lambda_n s} = \sum_{n=1}^{\infty}A_n(e^{-\lambda_n s}-e^{-\lambda_{n+1} s})$$

$$= \sum_{n=1}^{\infty}A_n s\int\limits_{\lambda_n}^{\lambda_{n+1}} e^{-sy}\,dy = s\int\limits_{0}^{\infty}A(y)e^{-sy}\,dy.$$

Since $A(y)$ is of bounded variation in any finite interval, the result follows from Theorem 24.

1.27. Fourier's theorem for analytic functions. The following form of Fourier's theorem applies to a class of analytic functions.

THEOREM 26. *Let $f(z)$ be an analytic function, regular for*

$$-a < y < b,$$

where $a > 0$, $b > 0$. In any strip interior to $-a < y < b$, let

$$f(z) = \begin{cases} O(e^{-(\lambda-\epsilon)x}) & (x \to \infty) \\ O(e^{(\mu-\epsilon)x}) & (x \to -\infty) \end{cases} \tag{1.27.1}$$

for every positive ϵ, where $\lambda > 0$, $\mu > 0$. Then $F(w)$, defined by (1.2.5), *satisfies conditions similar to those imposed on $f(z)$, with a, b, λ, μ replaced by λ, μ, b, a; and*

$$f(z) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}F(w)e^{-izw}\,dw \tag{1.27.2}$$

for every z in the strip $-a < y < b$.

We have $$F(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}f(\zeta)e^{i\zeta w}\,d\zeta,$$

and the integral converges uniformly for $-\lambda < v < \mu$. Hence $F(w)$ is analytic in this strip. By an obvious application of Cauchy's theorem we may take the integral along any line of the strip parallel to the real axis. Thus

$$F(w) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(\xi+i\eta) e^{i(\xi+i\eta)(u+iv)}\, d\xi = O(e^{-\eta u}),$$

and by taking η arbitrarily near to $-a$ or b the order-results for $F(w)$ follow.

The reciprocal formula (1.27.2) can be deduced from Theorem 23; it can also be proved directly by the theorem of residues. Let $--a < -\alpha < y < \beta < b$. Then

$$\begin{aligned} \frac{1}{\sqrt{(2\pi)}} \int_{0}^{\infty} F(w) e^{-izw}\, dw &= \frac{1}{2\pi} \int_{0}^{\infty} e^{-izw}\, dw \int_{i\beta-\infty}^{i\beta+\infty} f(\zeta) e^{i\zeta w}\, d\zeta \\ &= \frac{1}{2\pi} \int_{i\beta-\infty}^{i\beta+\infty} f(\zeta)\, d\zeta \int_{0}^{\infty} e^{-i(z-\zeta)w}\, dw \\ &= \frac{1}{2\pi i} \int_{i\beta-\infty}^{i\beta+\infty} \frac{f(\zeta)}{z-\zeta}\, d\zeta, \end{aligned}$$

the inversion being justified by absolute convergence. Similarly,

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{0} F(w) e^{-izw}\, dw = \frac{1}{2\pi i} \int_{-i\alpha-\infty}^{-i\alpha+\infty} \frac{f(\zeta)}{\zeta-z}\, d\zeta,$$

and, by an obvious application of the theorem of residues, the sum of the right-hand sides is $f(z)$.

1.28. Summability of the complex form.

The various summability theorems have obvious extensions to the complex form of the theorem. It will be sufficient to state one of them.

THEOREM 27. *Let $f(t)$ belong to $L(-\infty,\infty)$, or, more generally, let*

$$\int_{\to-\infty}^{\to\infty} f(t) e^{iut}\, dt$$

converge uniformly in any finite interval. Then

$$\frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) e^{-ixu}\, du \int_{\to-\infty}^{\to\infty} f(t) e^{iut}\, dt$$

is equal to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ *wherever this expression has a meaning; to* $f(x)$ *wherever* $f(x)$ *is continuous; and to* $f(x)$ *for almost all values of* x.

On inverting the order of integration the integral reduces to (1.16.1), and the result then follows from Theorems 14 and 20.

1.29. Mellin's inversion formula. Theorems on Mellin's formula may be obtained from theorems on Fourier's formula by a change of variable, as the formula itself was obtained in § 1.5; and of course there is no difficulty in adapting the arguments to give a direct proof in each case.

We shall state only the most important theorems.

THEOREM 28. *Let* $y^{k-1}f(y)$ *belong to* $L(0,\infty)$, *and let* $f(y)$ *be of bounded variation in the neighbourhood of the point* $y = x$. *Let*

$$\mathfrak{F}(s) = \int_0^\infty f(x)x^{s-1}\,dx \quad (s = k+it). \tag{1.29.1}$$

Then
$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{k-iT}^{k+iT}\mathfrak{F}(s)x^{-s}\,ds. \tag{1.29.2}$$

THEOREM 29. *Let* $\mathfrak{F}(k+iu)$ *belong to* $L(-\infty,\infty)$, *and let it be of bounded variation in the neighbourhood of the point* $u = t$. *Let*

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)x^{-s}\,ds. \tag{1.29.3}$$

Then
$$\tfrac{1}{2}[\mathfrak{F}\{k+i(t+0)\}+\mathfrak{F}\{k+i(t-0)\}] = \lim_{\lambda\to\infty}\int_{1/\lambda}^{\lambda} f(x)x^{k+it-1}\,dx. \tag{1.29.4}$$

Both theorems are obtained by changes of the variable in Theorem 23.

In some examples the following theorem is required.

THEOREM 30. *Let*

$$\mathfrak{F}(k+it) = \phi(t)e^{i\psi(t)},$$

where $\phi(t)$ *and* $\psi(t)$ *satisfy the conditions of Theorem* 11, *both as* $t\to\infty$ *and* $t\to-\infty$; *or let*

$$e^{kx}f(e^x) = \phi(x)e^{i\psi(x)},$$

where ϕ *and* ψ *satisfy such conditions. Then Mellin's formulae hold, the integrals being non-absolutely convergent.*

This follows from Theorem 11 by the usual substitutions.

THEOREM 31. *Let* $f(z)$ *be an analytic function of* $z = re^{i\theta}$, *regular for* $-\alpha < \theta < \beta$, *where* $0 < \alpha \leqslant \pi$, $0 < \beta \leqslant \pi$; *and let* $f(z)$ *be* $O(|z|^{-a-\epsilon})$ *for small* z, *and* $O(|z|^{-b+\epsilon})$ *for large* z, *where* $a < b$, *uniformly in any angle interior to the above.*

Then $\mathfrak{F}(s)$, *defined by* (1.29.1), *is an analytic function of* s, *regular for* $a < \sigma < b$; *and*

$$\mathfrak{F}(s) = \begin{cases} O(e^{-(\beta-\epsilon)t}) & (t \to \infty) \\ O(e^{(\alpha-\epsilon)t}) & (t \to -\infty) \end{cases}$$

for every positive ϵ, *uniformly in any strip interior to* $a < \sigma < b$; *and* (1.29.3) *holds for* $a < k < b$.

Conversely, if $\mathfrak{F}(s)$ *is a given function satisfying the above conditions and* $f(x)$ *is defined by* (1.29.3), *then* $f(x)$ *satisfies the conditions previously imposed on it, and* (1.29.1) *holds.*

This follows from Theorem 26, or it may easily be proved by an analogous argument.

THEOREM 32. *Let* $f(x)x^{k-1}$ *be* $L(0,\infty)$; *or, more generally, let*

$$\int_{\to 0}^{\to\infty} f(x)x^{s-1}\,dx = \mathfrak{F}(s) \tag{1.29.5}$$

be uniformly convergent for $s = k+it$, t *in any finite interval. Then*

$$\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{k-i\lambda}^{k+i\lambda}\left(1-\frac{|t|}{\lambda}\right)\mathfrak{F}(s)x^{-s}\,ds$$

is equal to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ *wherever this expression has a meaning, and in particular to* $f(x)$ *wherever* $f(x)$ *is continuous; and to* $f(x)$ *for almost all* x.

In the inverse form the assumption is that $\mathfrak{F}(k+it)$ *is* L, *and the conclusion*

$$\lim_{\mu\to\infty}\int_{1/\mu}^{\mu}\left(1-\frac{|\log x|}{\log\mu}\right)f(x)x^{s-1}\,dx = \mathfrak{F}(s)$$

almost everywhere.

This follows from Theorem 27 by the usual changes of variable.

A particular case[†] is that, if

$$\int_{\to 0} f(x)x^{a-1}\,dx, \qquad \int^{\to\infty} f(x)x^{b-1}\,dx,$$

where $a < b$, converge, then the result holds for $a < k < b$; for then

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[†] Hardy (8).

(1.29.5) converges uniformly in any finite region interior to the strip $a < \sigma < b$. In this case $\mathfrak{F}(s)$ is analytic in the strip.

1.30. The Laplace formulae.

The simplest theorem on the formula (1.4.3) is

THEOREM 33. *A necessary and sufficient condition that $f(z)$ should be of the form*

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \phi(w)e^{zw}\,dw, \tag{1.30.1}$$

where Γ is a closed contour surrounding the origin, is that it should be an integral function of exponential type, i.e. such that $f(z) = O(e^{c|z|})$ for some c.

The formula (1.30.1) plainly defines an integral function of z; and, if $|w| \leqslant c$ on the contour, $f(z) = O(e^{c|z|})$. Hence the condition is necessary.

Conversely, suppose that it is satisfied, and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then by Cauchy's inequality

$$|a_n| \leqslant \frac{1}{r^n}\max_{|z|=r}|f(z)| < \frac{Ke^{cr}}{r^n}$$

for all values of r. Taking $r = n$,

$$|a_n| < Ke^{cn}n^{-n}.$$

Hence the series
$$\phi(w) = \sum_{n=0}^{\infty}\frac{n!\,a_n}{w^{n+1}}$$

is convergent if w is sufficiently large, say for $|w| > M$. Let Γ be a simple closed curve surrounding the origin, and lying entirely outside the circle $|w| = M$. Then by uniform convergence

$$\frac{1}{2\pi i}\int_{\Gamma}\phi(w)e^{zw}\,dw = \sum_{n=0}^{\infty}\frac{n!\,a_n}{2\pi i}\int_{\Gamma}\frac{e^{zw}}{w^{n+1}}\,dw = \sum_{n=0}^{\infty} a_n z^n = f(z),$$

the required result.

The reciprocal formula is

$$\phi(w) = \int_0^{\infty} f(x)e^{-xw}\,dx$$

as in § 1.4, but in general this holds for $\mathbf{R}(w) > c$ only.

For $f(z)$ in (1.30.1) to vanish identically, it is plainly not necessary for $\phi(w)$ to vanish; it is sufficient for $\phi(w)$ to be regular within Γ. Hence, if we are given $f(z)$ and Γ, (1.30.1) does not determine $\phi(w)$ uniquely. It does so, however, if $\phi(w)$ is given to be regular and zero at infinity; and there is a more general result of the same kind.

THEOREM 34. *Let $\phi(w)$ be regular for sufficiently large w, except for a pole of order n at infinity, and let*

$$\int\limits_{\Gamma} \phi(w)e^{wt}\,dw = 0$$

for all t, Γ being a simple closed contour surrounding the origin. Then

$$\phi(w) = a_0+a_1 w+\ldots+a_n w^n.$$

Let
$$\psi(w) = \phi(w)-a_0-\ldots-a_n w^n,$$

where $a_0+\ldots+a_n w^n$ is the principal part of $\phi(w)$ at infinity. Then

$$\int\limits_{\Gamma} \psi(w)e^{wt}\,dw = 0,$$

and $\psi(w) \to 0$ as $|w| \to \infty$.

Multiply by e^{-zt}, where $\mathbf{R}(z) > \max\limits_{\Gamma} \mathbf{R}(w)$, and integrate over $(0,\infty)$. We obtain

$$\int\limits_{\Gamma} \psi(w)\frac{1}{z-w}\,dw = 0,$$

and this holds by analytic continuation for any z outside Γ.

Hence, by the calculus of residues, if Γ' is a circle of radius $R > |z|$,

$$\psi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma'} \frac{\psi(w)}{w-z}\,dw,$$

and, making $R \to \infty$, the right-hand side tends to 0. Hence $\psi(z) = 0$.

II

AUXILIARY FORMULAE

2.1. Formalities. If $F(x)$ and $G(x)$ are the Fourier transforms of $f(x)$ and $g(x)$, we have formally

$$\int_{-\infty}^{\infty} F(x)G(x)\,dx = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} G(x)\,dx \int_{-\infty}^{\infty} f(t)e^{ixt}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} G(x)e^{ixt}\,dx = \int_{-\infty}^{\infty} f(t)g(-t)\,dt. \tag{2.1.1}$$

If $g(t)$ is replaced by $\bar{g}(-t)$, $G(x)$ is replaced by $\bar{G}(x)$, so that an equivalent formula is

$$\int_{-\infty}^{\infty} F(x)\bar{G}(x)\,dx = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx. \tag{2.1.2}$$

If $g = f$, we obtain

$$\int_{-\infty}^{\infty} |F(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx. \tag{2.1.3}$$

For even functions the formulae reduce to

$$\int_{0}^{\infty} F_c(x)G_c(x)\,dx = \int_{0}^{\infty} f(x)g(x)\,dx, \tag{2.1.4}$$

and

$$\int_{0}^{\infty} \{F_c(x)\}^2\,dx = \int_{0}^{\infty} \{f(x)\}^2\,dx; \tag{2.1.5}$$

for odd functions they reduce to

$$\int_{0}^{\infty} F_s(x)G_s(x)\,dx = \int_{0}^{\infty} f(x)g(x)\,dx \tag{2.1.6}$$

and

$$\int_{0}^{\infty} \{F_s(x)\}^2\,dx = \int_{0}^{\infty} \{f(x)\}^2\,dx. \tag{2.1.7}$$

These formulae are analogous to Parseval's formula

$$\frac{1}{\pi}\int_{0}^{2\pi} \{f(x)\}^2\,dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2+b_n^2)$$

in the theory of Fourier series. They will be known generally as the Parseval formulae.† Again,

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} F(t)G(t)e^{-ixt}\,dt = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(t)e^{-ixt}\,dt \int\limits_{-\infty}^{\infty} g(u)e^{itu}\,du$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g(u)\,du \int\limits_{-\infty}^{\infty} F(t)e^{-it(x-u)}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} g(u)f(x-u)\,du. \tag{2.1.8}$$

Thus the functions

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} g(u)f(x-u)\,du, \qquad F(x)G(x), \tag{2.1.9}$$

are Fourier transforms. The integral obtained is called the *resultant* of $f(x)$ and $g(x)$.

The process may clearly be repeated. The functions

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} h(v)\,dv \int\limits_{-\infty}^{\infty} g(u)f(x-u-v)\,du, \qquad F(x)G(x)H(x) \tag{2.1.10}$$

are transforms. So generally are

$$\left.\begin{aligned} &\frac{1}{(2\pi)^{\frac{1}{2}n}}\int\limits_{-\infty}^{\infty} f_n(u_n)\,du_n \int\limits_{-\infty}^{\infty} f_{n-1}(u_{n-1})\,du_{n-1}\dots\times \\ &\qquad\times\int\limits_{-\infty}^{\infty} f_1(u_1)f(x-u_1-\dots-u_n)\,du_1 \\ &\text{and}\qquad F(x)F_1(x)\dots F_n(x). \end{aligned}\right\} \tag{2.1.11}$$

There are analogous formulae for Mellin transforms, which may be obtained by transformation from the above, or directly as follows. If $\mathfrak{F}(s)$, $\mathfrak{G}(s)$ are the Mellin transforms of $f(x)$ and $g(x)$,

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\mathfrak{G}(1-s)\,ds = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \mathfrak{G}(1-s)\,ds \int\limits_{0}^{\infty} f(x)x^{s-1}\,dx$$

$$= \frac{1}{2\pi i}\int\limits_{0}^{\infty} f(x)\,dx \int\limits_{k-i\infty}^{k+i\infty} \mathfrak{G}(1-s)x^{s-1}\,ds = \int\limits_{0}^{\infty} f(x)g(x)\,dx, \tag{2.1.12}$$

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† The earliest reference to the formulae of which I know is in Rayleigh (1). See also Hardy (3–5), Ramanujan (1).

or, alternatively,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\mathfrak{G}(1-s)\,ds = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\,ds\int_0^\infty g(x)x^{-s}\,dx$$

$$= \frac{1}{2\pi i}\int_0^\infty g(x)\,dx\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)x^{-s}\,ds = \int_0^\infty g(x)f(x)\,dx.$$

Similarly,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\mathfrak{G}(s)\,ds = \frac{1}{2\pi i}\int_0^\infty g(x)\,dx\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)x^{s-1}\,ds$$

$$= \int_0^\infty g(x)f\left(\frac{1}{x}\right)\frac{dx}{x}. \tag{2.1.13}$$

If $g = f$, and both are real, (2.1.12) with $k = \frac{1}{2}$ gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathfrak{F}(\tfrac{1}{2}+it)|^2\,dt = \int_0^\infty \{f(x)\}^2\,dx. \tag{2.1.14}$$

Also

$$\int_0^\infty f(x)g(x)x^{s-1}\,dx = \frac{1}{2\pi i}\int_0^\infty g(x)x^{s-1}\,dx\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(w)x^{-w}\,dw$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(w)\,dw\int_0^\infty g(x)x^{s-w-1}\,dx$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(w)\mathfrak{G}(s-w)\,dw, \tag{2.1.15}$$

i.e.

$$f(x)g(x), \qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(w)\mathfrak{G}(s-w)\,dw \tag{2.1.16}$$

are Mellin transforms.

Again,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\mathfrak{G}(s)x^{-s}\,ds = \frac{1}{2\pi i}\int_0^\infty g(u)\,du\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)u^{s-1}x^{-s}\,ds$$

$$= \int_0^\infty g(u)f\left(\frac{x}{u}\right)\frac{du}{u}, \tag{2.1.17}$$

another sort of resultant.'

Hence we obtain as Mellin transforms

$$\int_0^\infty g(u)f\left(\frac{x}{u}\right)\frac{du}{u}, \qquad \mathfrak{F}(s)\mathfrak{G}(s). \tag{2.1.18}$$

Repeating the process, we obtain as Mellin transforms

$$\int_0^\infty f_n(u_n)\frac{du_n}{u_n}\int_0^\infty f_{n-1}(u_{n-1})\frac{du_{n-1}}{u_{n-1}}\cdots\int_0^\infty f_1(u_1)f\left(\frac{x}{u_1 u_2 \ldots u_n}\right)\frac{du_1}{u_1},$$

$$\mathfrak{F}(s)\mathfrak{F}_1(s)\ldots\mathfrak{F}_n(s). \tag{2.1.19}$$

From the Laplace integral formulae we derive similarly the formulae

$$\begin{aligned}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\phi(s)\psi(s)e^{sx}\,ds &= \frac{1}{2\pi i}\int\psi(s)e^{sx}\,ds\int_0^\infty f(y)e^{-sy}\,dy\\ &= \frac{1}{2\pi i}\int_0^\infty f(y)\,dy\int\psi(s)e^{s(x-y)}\,ds\\ &= \int_0^x f(y)g(x-y)\,dy,\end{aligned} \tag{2.1.20}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int\phi(s)\psi(-s)e^{sx}\,ds &= \frac{1}{2\pi i}\int_0^\infty f(y)\,dy\int\psi(-s)e^{s(x-y)}\,ds\\ &= \int_{\max(x,0)}^\infty f(y)g(y-x)\,dy.\end{aligned} \tag{2.1.21}$$

We can also introduce parameters into the formulae without altering them essentially. Since

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^\infty f(ay)e^{ixy}\,dy = \frac{1}{a\sqrt{(2\pi)}}\int_{-\infty}^\infty f(u)e^{ixu/a}\,du = \frac{1}{a}F\left(\frac{x}{a}\right),$$

the transform of $f(ay)$ is $\frac{1}{a}F\left(\frac{x}{a}\right)$. Thus e.g.

$$\int_{-\infty}^\infty f(at)g(-bt)\,dt = \frac{1}{ab}\int_{-\infty}^\infty F\left(\frac{x}{a}\right)G\left(\frac{x}{b}\right)dx. \tag{2.1.22}$$

Similar changes may be made in the other formulae; e.g.

$$\int_0^\infty f(ax)x^{s-1}\,dx = a^{-s}\int_0^\infty f(\xi)\xi^{s-1}\,d\xi,$$

so that the Mellin transform of $f(ax)$ is $a^{-s}\mathfrak{F}(s)$. Thus

$$\int_0^\infty f(ax)g(bx)\,dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\mathfrak{G}(1-s)a^{-s}b^{s-1}\,ds, \tag{2.1.23}$$

and similarly in the other formulae.

2.2. Conditions for validity. We shall now give some sets of conditions for the validity of the above formulae. Some of the most important conditions depend on the theory of mean convergence, and must be postponed until later chapters. The conditions which we give here depend on analysis resembling that of Chapter I.

We begin with (2.1.1) and its special cases.

THEOREM 35. *If $f(x)$ and $G(x)$ belong to $L(-\infty,\infty)$, and $F(x)$ and $g(x)$ are their transforms, then* (2.1.1) *holds.*

For the inversions used in obtaining (2.1.1) are justified by absolute convergence. The theorem implies that f and G are the given functions, and F and g defined in terms of them.

The theorem of course includes the corresponding theorems for cosine and sine transforms.

It follows also that, if $f(x)$ and $g(x)$ are $L(-\infty,\infty)$, and $G(x)$, defined as the transform of $g(x)$, is $L(-\infty,\infty)$, then (2.1.1) holds. For, by Theorem 27, $g(x)$ is the transform of $G(x)$.

2.3. We next take some cases of Parseval's formula suggested by Theorem 6. Here the conditions are more appropriate to the half-line $(0,\infty)$, and we consider cosine and sine transforms separately.

THEOREM 36.† *Let $f(x)$ belong to $L(0,\infty)$, and, in some interval ending at* 0, *tend steadily to a limit as $x\to 0$. Let $g(x)$ be the cosine transform of $G_c(x)$, which is integrable over any finite interval, and tends steadily to* 0 *as $x\to\infty$. Then*

$$\int_0^{\to\infty} F_c(x)G_c(x)\,dx = \int_{\to 0}^{\infty} f(x)g(x)\,dx. \tag{2.3.1}$$

We have to justify the inversion

$$\int_0^{\to\infty} G_c(y)\,dy \int_0^\infty f(x)\cos xy\,dx = \int_{\to 0}^{\infty} f(x)\,dx \int_0^{\to\infty} G_c(y)\cos xy\,dy. \tag{2.3.2}$$

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† Hardy (5).

Now

$$\int\limits_0^\lambda G_c(y)\,dy\int\limits_\delta^\infty f(x)\cos xy\,dx = \int\limits_\delta^\infty f(x)\,dx\int\limits_0^\lambda G_c(y)\cos xy\,dy, \tag{2.3.3}$$

for every finite λ, by uniform convergence. By the second mean-value theorem

$$\int\limits_\lambda^{\to\infty} G_c(y)\cos xy\,dy = G_c(\lambda)\int\limits_\lambda^{\lambda'}\cos xy\,dy = O\left\{\frac{G_c(\lambda)}{x}\right\}.$$

Hence

$$\lim_{\lambda\to\infty}\int\limits_\delta^\infty f(x)\,dx\int\limits_\lambda^{\to\infty} G_c(y)\cos xy\,dy = 0, \tag{2.3.4}$$

and (2.3.3), (2.3.4) give

$$\int\limits_0^{\to\infty} G_c(y)\,dy\int\limits_\delta^\infty f(x)\cos xy\,dx = \int\limits_\delta^\infty f(x)\,dx\int\limits_0^{\to\infty} G_c(y)\cos xy\,dy \tag{2.3.5}$$

for every $\delta > 0$. It is now sufficient to prove that

$$\lim_{\delta\to 0}\int\limits_0^{\to\infty} G_c(y)\,dy\int\limits_0^\delta f(x)\cos xy\,dx = 0. \tag{2.3.6}$$

If, e.g., $f(x)$ is steadily decreasing in $(0,\delta)$,

$$\begin{aligned}\int\limits_{y_1}^{y_2} G_c(y)\,dy\int\limits_0^\delta f(x)\cos xy\,dx &= \int\limits_0^\delta f(x)\,dx\int\limits_{y_1}^{y_2} G_c(y)\cos xy\,dy\\ &= f(+0)\int\limits_0^\xi dx\int\limits_{y_1}^{y_2} G_c(y)\cos xy\,dy\\ &= f(+0)\int\limits_{y_1}^{y_2} G_c(y)\frac{\sin\xi y}{y}\,dy,\end{aligned}$$

where $0 < \xi < \delta$; and

$$\int\limits_Y^{y_2} G_c(y)\frac{\sin\xi y}{y}\,dy = G_c(Y)\int\limits_Y^{\eta}\frac{\sin\xi y}{y}\,dy = O\{G_c(Y)\}$$

for all ξ, while, for a fixed Y,

$$\int\limits_{y_1}^{Y} G_c(y)\frac{\sin\xi y}{y}\,dy \to 0$$

as $\xi \to 0$. The result therefore follows on choosing first Y sufficiently large and then δ sufficiently small.

THEOREM 37. *The corresponding theorem for sine transforms holds provided that, in addition,* $G_s(x)/x$ *belongs to* $L(1,\infty)$.

In this case we encounter

$$\int_{v_1}^{v_2} G_s(y)\frac{1-\cos\xi y}{y}\,dy$$

at the last stage of the proof, and the extra condition is required here.

2.4. In the above theorems the functions on which the conditions bear are on opposite sides of the Parseval formula. We next prove a theorem in which they are on the same side.†

THEOREM 38. *Let* $f(x)$ *belong to* $L(0,\infty)$. *Let* $g(x)$ *be positive, non-increasing, and tend to* 0 *as* $x\to\infty$, *and let*

$$\int_0^1 \frac{|f(t)|}{t}\,dt\int_0^t g(u)\,du<\infty. \tag{2.4.2}$$

Then

$$\int_{\to 0}^{\to\infty} F_c(x)G_c(x)\,dx=\int_0^\infty f(x)g(x)\,dx, \tag{2.4.3}$$

and similarly for sine transforms.

We have

$$\int_\xi^X F_c(x)G_c(x)\,dx=\sqrt{\left(\frac{2}{\pi}\right)}\int_\xi^X G_c(x)\,dx\int_0^\infty f(t)\cos xt\,dt$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(t)\,dt\int_\xi^X G_c(x)\cos xt\,dx$$

$$=\frac{2}{\pi}\int_0^\infty f(t)\,dt\int_\xi^X \cos xt\,dx\int_0^{\to\infty} g(u)\cos xu\,du$$

$$=\frac{2}{\pi}\int_0^\infty f(t)\,dt\int_0^{\to\infty} g(u)\,du\int_\xi^X \cos xt\cos xu\,dx$$

$$=\int_0^\infty f(t)\{g(t,X)-g(t,\xi)+g(-t,X)-g(-t,\xi)\}\,dt, \tag{2.4.5}$$

where

$$g(t,x)=\frac{1}{\pi}\int_0^{\to\infty} g(u)\frac{\sin x(u-t)}{u-t}\,du.$$

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† See Hardy and Titchmarsh (5).

The inversion of the x and t integrations is justified by the uniform convergence of the t-integral, and that of the x and u integrations by that of the u-integral.

Since $g(t)$ is non-increasing, $g(t,X) \to g(t)$ as $X \to \infty$ for almost all positive t, and $g(-t,X) \to 0$. Also as $\xi \to 0$

$$g(t,\xi) = \int\limits_0^U g(u)\frac{\sin\xi(u-t)}{u-t}\,du + O\{g(U)\}$$

$$= O\left(\xi\int\limits_0^U g(u)\,du\right) + O\{g(U)\} = o(1)$$

by choosing first U and then ξ.

Now

$$\left|\int\limits_{\frac12 t}^{\to\infty} g(u)\frac{\sin x(u-t)}{u-t}\,du\right| = \left|g(\tfrac12 t+0)\int\limits_{\frac12 t}^{T}\frac{\sin x(u-t)}{u-t}\,du\right|$$

$$< Ag(\tfrac12 t) < \frac{A}{t}\int\limits_0^{\frac12 t} g(u)\,du < \frac{A}{t}\int\limits_0^{t} g(u)\,du,$$

and
$$\left|\int\limits_0^{\frac12 t} g(u)\frac{\sin x(u-t)}{u-t}\,du\right| \leqslant \int\limits_0^{\frac12 t}\frac{g(u)}{t-u}\,du \leqslant \frac{2}{t}\int\limits_0^{\frac12 t} g(u)\,du$$

$$\leqslant \frac{2}{t}\int\limits_0^{t} g(u)\,du.$$

Hence
$$|f(t)g(t,x)| \leqslant A\,\frac{|f(t)|}{t}\int\limits_0^{t} g(u)\,du,$$

which belongs to $L(0,1)$, by hypothesis; and it belongs to $L(1,\infty)$, since $f(t)$ belongs to $L(1,\infty)$, and

$$\frac{1}{t}\int\limits_0^{t} g(u)\,du \to 0.$$

The result now follows from (2.4.5) on making $X \to \infty$, $\xi \to 0$, by dominated convergence.

Immediate corollaries are:

(i) If $f(x)$ belongs to $L(0,\infty)$, and $g(x)$ is of bounded variation in $(0,\infty)$, and tends to 0 at infinity, then (2.4.3) holds.

(ii) If f is L and g bounded in $(0, 1)$, then (2.4.2) holds and the theorem follows.

(iii) If f is bounded and $g(x)\log(1/x)$ is L in $(0,1)$, (2.4.2) holds and the theorem follows.

Apparently f bounded and g L in $(0,1)$ is not sufficient.

2.5. In a later chapter we find some examples of Parseval's formula which evade all the above theorems. These are cases where the existence of the transforms and the convergence of the integrals involved is obvious enough, and all that is needed is to prove the equality of the two sides of Parseval's formula. We can deal with some such cases by means of the following theorem.

THEOREM 39. *Let $f(x)$ and $g(x)$ be integrable over any finite interval. Let*

$$F(x,a)=\frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a}f(t)e^{ixt}\,dt,\qquad G(x,a)=\frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a}g(t)e^{ixt}\,dt,$$

and
$$\chi(x,a,b)=\int_{-a}^{b}g(u)f(x-u)\,du$$

be all $O(e^{c|x|})$ for some positive c, independently of a and b, and tend to $F(x)$, $G(x)$, and $\chi(x)$ as $a\to\infty$, $b\to\infty$, for almost all x. Then

$$\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}F(x)G(x)\,dx=\tfrac12\{\chi(+0)+\chi(-0)\}$$

provided that the limits indicated exist.

Let $\lambda>0$. Then by dominated convergence

$$\int_{-\infty}^{\infty}F(t)e^{-\frac14t^2/\lambda+iut}\,dt=\lim_{a\to\infty}\int_{-\infty}^{\infty}F(t,a)e^{-\frac14t^2/\lambda+iut}\,dt$$
$$=\lim_{a\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a}f(x)\,dx\int_{-\infty}^{\infty}e^{-\frac14t^2/\lambda+iut+ixt}\,dt$$
$$=\lim_{a\to\infty}\sqrt{(2\lambda)}\int_{-a}^{a}f(x)e^{-\lambda(x+u)^2}\,dx,$$

and the convergence is uniform over a finite u-interval. Hence

$$\int_{-b}^{b}g(u)\,du\int_{-\infty}^{\infty}F(t)e^{-\frac14t^2/\lambda+iut}\,dt=\lim_{a\to\infty}\sqrt{(2\lambda)}\int_{-b}^{b}g(u)\,du\int_{-a}^{a}f(x)e^{-\lambda(x+u)^2}\,dx$$
$$=\lim_{a\to\infty}\sqrt{(2\lambda)}\int_{-b}^{b}g(u)\,du\int_{-a+u}^{a+u}f(x-u)e^{-\lambda x^2}\,dx$$

$$= \lim_{a\to\infty} \sqrt{(2\lambda)} \int_{-a-b}^{a+b} e^{-\lambda x^2}\,dx \int_{\max(x-a,\,-b)}^{\min(x+a,\,b)} g(u)f(x-u)\,du$$

$$= \sqrt{(2\lambda)} \int_{-\infty}^{\infty} e^{-\lambda x^2}\chi(x,b,b)\,dx,$$

by dominated convergence. Also we may invert the left-hand side, by uniform convergence, and obtain

$$\sqrt{(2\pi)} \int_{-\infty}^{\infty} F(t)G(t,b)e^{-\frac{1}{4}t^2/\lambda}\,dt.$$

Hence, making $b \to \infty$, and using dominated convergence,

$$\int_{-\infty}^{\infty} F(t)G(t)e^{-\frac{1}{4}t^2/\lambda}\,dt = \sqrt{\left(\frac{\lambda}{\pi}\right)} \int_{-\infty}^{\infty} e^{-\lambda x^2}\chi(x)\,dx.$$

Making $\lambda \to \infty$, the result now follows from Theorem 16.

In particular, the result holds if f and g belong to $L(-\infty,\infty)$, and one of them is bounded.

2.6. Transform of a resultant. We now turn to (2.1.8), giving the Fourier transform of a product, or of a resultant. From one point of view this is merely a case of Parseval's formula, since $f(x-u)$ is the transform of $F(t)e^{-ixt}$. A new problem arises, however, when we consider all values of x at once. We then ask whether (2.1.9) are transforms belonging to one of the general classes already considered.

THEOREM 40. *Let $f(x)$ be the transform of a function $F(x)$ of $L(-\infty,\infty)$, and let $g(x)$ belong to $L(-\infty,\infty)$ (so that its transform $G(x)$ is bounded). Then $\sqrt{(2\pi)}F(x)G(x)$ belongs to $L(-\infty,\infty)$, and its transform is*

$$k(x) = \int_{-\infty}^{\infty} g(u)f(x-u)\,du.$$

For the inversion in (2.1.8) is justified by absolute convergence.

THEOREM 41. *Let $f(x)$ and $g(x)$ belong to $L(-\infty,\infty)$. Then so does $k(x)$, and its transform is $\sqrt{(2\pi)}F(x)G(x)$.*

For
$$\int_{-a}^{a} k(u)e^{ixu}\,du = \int_{-a}^{a} e^{ixu}\,du \int_{-\infty}^{\infty} f(v)g(u-v)\,dv$$

$$= \int_{-\infty}^{\infty} f(v)\,dv \int_{-a}^{a} g(u-v)e^{iux}\,du$$

$$= \int_{-\infty}^{\infty} f(v)e^{ivx}\,dv \int_{-a-v}^{a-v} g(t)e^{ixt}\,dt.$$

The inner integral converges boundedly to $\sqrt{(2\pi)}G(x)$. Hence

$$\int_{-\infty}^{\infty} k(u)e^{ixu}\,du = \sqrt{(2\pi)}\int_{-\infty}^{\infty} f(v)e^{ivx}G(x)\,dv = 2\pi F(x)G(x).$$

2.7. Mellin transforms.

THEOREM 42. *Let* $x^{k-1}f(x)$ *be* $L(0,\infty)$, *and* $\mathfrak{G}(1-k-it)$ *be* $L(-\infty,\infty)$, *or alternatively let* $\mathfrak{F}(k+it)$ *be* $L(-\infty,\infty)$, *and* $x^{-k}g(x)$ *be* $L(0,\infty)$. *Then* (2.1.12) *holds.*

For the inversion which gives the formula is justified by absolute convergence.

THEOREM 43. *Let* $f(x)$ *and* $g(x)$ *be integrable over any finite interval not ending at* $x=0$. *Let*

$$\mathfrak{F}(s,a) = \int_{1/a}^{a} f(x)x^{s-1}\,dx, \qquad \mathfrak{G}(s,a) = \int_{1/a}^{a} g(x)x^{s-1}\,dx$$

tend to $\mathfrak{F}(s)$, $\mathfrak{G}(s)$ *for* $\sigma = k$, $\sigma = 1-k$ *respectively, for almost all* t, *in such a way that* $e^{-c|s|}\mathfrak{F}(s,a)$, $e^{-c|s|}\mathfrak{G}(s,a)$ *are, for some positive* c, *bounded independently of* a. *Let*

$$\frac{\xi^{1-k}}{\xi^{c}+\xi^{-c}}\int_{a}^{b} f(x)g(\xi x)\,dx.$$

be bounded for all a, b, ξ, *and, as* $a\to 0$, $b\to\infty$, *converge to a continuous limit in the neighbourhood of* $\xi = 1$. *Then*

$$\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{k-i\lambda}^{k+i\lambda} \mathfrak{F}(s)\mathfrak{G}(1-s)\,ds = \int_{\to 0}^{\to\infty} f(x)g(x)\,dx,$$

provided the left-hand side exists.

This follows by a change of variable from Theorem 39.

The analogue of Theorem 41 is

THEOREM 44. *Let* $x^{k}f(x)$ *and* $x^{k}g(x)$ *belong to* $L(0,\infty)$, *and let*

$$h(x) = \int_{0}^{\infty} f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}.$$

Then $x^{k}h(x)$ *belongs to* $L(0,\infty)$, *and its Mellin transform is* $\mathfrak{F}(s)\mathfrak{G}(s)$, *with* $\sigma = k+1$.

2.8. Poisson's formula. This is

$$\sqrt{\beta}\left\{\tfrac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c(n\beta)\right\} = \sqrt{\alpha}\left\{\tfrac{1}{2}f(0)+\sum_{n=1}^{\infty}f(n\alpha)\right\}, \qquad (2.8.1)$$

where $\alpha\beta = 2\pi$, $\alpha > 0$.

We shall prove†

Theorem 45. *Let $f(x)$ be of bounded variation in* $(0,\infty)$, *and tend to* 0 *as* $x\to\infty$. *Then*

$$\sqrt{\beta}\sum_{n=1}^{\infty}F_c(n\beta)$$
$$=\sqrt{\alpha}\lim_{M\to\infty}\left[\tfrac{1}{2}f(0+0)+\sum_{m=1}^{M}\tfrac{1}{2}\{f(m\alpha-0)+f(m\alpha+0)\}-\frac{1}{\alpha}\int\limits_{0}^{(M+\frac{1}{2})\alpha}f(t)\,dt\right]. \tag{2.8.2}$$

If also $\int\limits_{0}^{\to\infty}f(t)\,dt$ *exists, then*

$$\sqrt{\beta}\Big\{\tfrac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c(n\beta)\Big\}$$
$$=\sqrt{\alpha}\Big[\tfrac{1}{2}f(0+0)+\sum_{m=1}^{\infty}\tfrac{1}{2}\{f(m\alpha-0)+f(m\alpha+0)\}\Big]. \tag{2.8.3}$$

If also $f(x)$ is continuous, then (2.8.1) *holds.*

Since $f(t)$ is the difference between two non-increasing functions, each of which $\to 0$ as $x\to\infty$, we may take it to be one such function.

The integral
$$F_c(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{0}^{\to\infty}f(t)\cos xt\,dt$$
exists for $x>0$, and

$$\sqrt{\beta}\sum_{m=1}^{n}F_c(m\beta)$$
$$=\sqrt{\left(\frac{2\beta}{\pi}\right)}\int\limits_{0}^{\to\infty}f(t)\sum_{m=1}^{n}\cos m\beta t=\sqrt{\left(\frac{2\beta}{\pi}\right)}\int\limits_{0}^{\to\infty}f(t)\left\{\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}-\tfrac{1}{2}\right\}dt,$$

which is the limit as $M\to\infty$ of

$$\sqrt{\left(\frac{2\beta}{\pi}\right)}\sum_{m=0}^{M}\int\limits_{2m\pi/\beta}^{(2m+1)\pi/\beta}\left\{f(t)-f\left(\frac{2m\pi}{\beta}+0\right)\right\}\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}\,dt+$$
$$+\sqrt{\left(\frac{2\beta}{\pi}\right)}\sum_{m=1}^{M}\int\limits_{(2m-1)\pi/\beta}^{2m\pi/\beta}\left\{f(t)-f\left(\frac{2m\pi}{\beta}-0\right)\right\}\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}\,dt+$$
$$+\sqrt{\left(\frac{\pi}{2\beta}\right)}\left[f(0+0)+\sum_{m=1}^{M}\left\{f\left(\frac{2m\pi}{\beta}+0\right)+f\left(\frac{2m\pi}{\beta}-0\right)\right\}\right]-$$
$$-\sqrt{\left(\frac{\beta}{2\pi}\right)}\int\limits_{0}^{(2M+1)\pi/\beta}f(t)\,dt. \tag{2.8.4}$$

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† I do not know whether this version of the theorem has been published previously. I obtained it by combining one of my own with one communicated to me by Dr. W. L. Ferrar. For other methods see Linfoot (1), Mordell (1).

Now

$$\int\limits_{2m\pi/\beta}^{(2m+1)\pi/\beta} \left\{f(t)-f\left(\frac{2m\pi}{\beta}+0\right)\right\}\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}\,dt$$

$$= \int\limits_{0}^{\pi/\beta} \left\{f\left(\frac{2m\pi}{\beta}+t\right)-f\left(\frac{2m\pi}{\beta}+0\right)\right\}\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}\,dt$$

$$= \left\{f\left(\frac{(2m+1)\pi}{\beta}-0\right)-f\left(\frac{2m\pi}{\beta}+0\right)\right\}\int\limits_{\xi}^{\pi/\beta}\frac{\sin(n+\frac{1}{2})\beta t}{2\sin\frac{1}{2}\beta t}\,dt,$$

by the second mean-value theorem. The last integral is bounded for all n and ξ, e.g. as

$$\int\limits_{\xi}^{\pi/\beta} \sin(n+\tfrac{1}{2})\beta t\left(\frac{1}{2\sin\frac{1}{2}\beta t}-\frac{1}{\beta t}\right)dt+\frac{1}{\beta}\int\limits_{(n+\frac{1}{2})\beta\xi}^{(n+\frac{1}{2})\pi}\frac{\sin u}{u}\,du,$$

and
$$\sum\left|f\left(\frac{(2m+1)\pi}{\beta}-0\right)-f\left(\frac{2m\pi}{\beta}+0\right)\right|$$

is convergent. Hence the first series on the right-hand side of (2.8.4) is convergent as $M\to\infty$, uniformly with respect to n; and each term tends to 0 as $n\to\infty$. Hence the limit of the sum is 0. Similarly for the second series. This proves (2.8.2); and (2.8.3) and (2.8.1) clearly follow from (2.8.2) in the cases stated.

There are also more complicated formulae of the same type. For example, Ramanujan† gives

$$\sqrt{\beta}\{F_c(\beta)-F_c(3\beta)-F_c(5\beta)+F_c(7\beta)+\ldots\}$$
$$=\sqrt{\alpha}\{f(\alpha)-f(3\alpha)-f(5\alpha)+\ldots\},$$

where $\alpha\beta=\frac{1}{4}\pi$; and

$$\sqrt{\beta}\{F_c(\beta)-F_c(5\beta)-F_c(7\beta)+F_c(11\beta)+F_c(13\beta)-\ldots\}$$
$$=\sqrt{\alpha}\{f(\alpha)-f(5\alpha)-\ldots\},$$

where $\alpha\beta=\frac{1}{6}\pi$, and 1, 5, 7, 11, 13,... are the numbers prime to 6. These formulae are easily verified by the above method.

2.9. There is another interesting formal method of procedure.‡ Suppose that $f(x)$ is represented by Mellin's integral

$$f(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\mathfrak{F}(s)x^{-s}\,ds.$$

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† Ramanujan (2).

‡ Ferrar (2).

Then formally

$$\sum_{n=1}^{\infty} f(n\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{F}(s)\sum_{n=1}^{\infty}(n\alpha)^{-s}\,ds \quad (c > 1)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{F}(s)\zeta(s)\alpha^{-s}\,ds.$$

Move the line of integration from $\sigma = c$ to $\sigma = -b$, where $b > 0$; $\zeta(s)$ has a simple pole at $s = 1$, with residue 1; and

$$\mathfrak{F}(s) = \frac{f(0)}{s} + \int_{0}^{1}\{f(x)-f(0)\}x^{s-1}\,dx + \int_{1}^{\infty} f(x)x^{s-1}\,dx$$

has in general a simple pole at $s = 0$, with residue $f(0)$. Since $\zeta(0) = -\frac{1}{2}$ we obtain

$$\tfrac{1}{2}f(0)+\sum_{n=1}^{\infty} f(n\alpha)-\alpha^{-1}\mathfrak{F}(1)$$

$$= \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} \mathfrak{F}(s)\zeta(s)\alpha^{-s}\,ds$$

$$= \frac{1}{2\pi i}\int_{1+b-i\infty}^{1+b+i\infty} \mathfrak{F}(1-s)\zeta(1-s)\alpha^{s-1}\,ds$$

$$= \frac{1}{\alpha\pi i}\int_{1+b-i\infty}^{1+b+i\infty} \mathfrak{F}(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\,\zeta(s)\left(\frac{2\pi}{\alpha}\right)^{-s} ds$$

$$= \frac{1}{\alpha\pi i}\sum_{n=1}^{\infty}\int_{1+b-i\infty}^{1+b+i\infty} \mathfrak{F}(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\left(\frac{2n\pi}{\alpha}\right)^{-s} ds.$$

But by (2.1.23), with f and g interchanged, and

$$g(x) = \cos x, \qquad \mathfrak{G}(s) = \Gamma(s)\cos\tfrac{1}{2}s\pi, \qquad b = 1,$$

$$F_c(a) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\,a^{-s}\,ds.$$

We have therefore obtained (2.8.1) again.

We shall not attempt to justify this process here. The main interest of it is that it suggests a method of dealing with sums such as

$$\sum_{n=1}^{\infty} d(n)f(n),$$

where $d(n)$ is the number of divisors of n. This sum, for example, gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathfrak{F}(s)\zeta^2(s)\,ds$$

$$=\tfrac{1}{4}f(0)+\mathfrak{F}'(1)-2\gamma\mathfrak{F}(1)+\frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty}\mathfrak{F}(s)\zeta^2(s)\,ds,$$

and the last term is

$$\frac{1}{2\pi i}\int_{1+b-i\infty}^{1+b+i\infty}\mathfrak{F}(1-s)\zeta^2(1-s)\,ds$$

$$=\frac{2}{\pi i}\int_{1+b-i\infty}^{1+b+i\infty}\mathfrak{F}(1-s)\Gamma^2(s)\cos^2\tfrac{1}{2}s\pi\,2^{-2s}\pi^{-2s}\zeta^2(s)\,ds$$

$$=\sum_{n=1}^{\infty}\frac{2d(n)}{\pi i}\int_{1+b-i\infty}^{1+b+i\infty}\mathfrak{F}(1-s)\Gamma^2(s)\cos^2\tfrac{1}{2}s\pi\,(4\pi^2n)^{-s}\,ds.$$

From (7.9.7) and (7.9.11) we deduce

$$\frac{2}{\pi}K_0(x)-Y_0(x)=\frac{1}{2\pi^2 i}\int_{k-i\infty}^{k+i\infty}\Gamma^2(\tfrac{1}{2}s)\cos^2\tfrac{1}{4}s\pi\,2^s x^{-s}\,ds\quad(k>1),$$

and, proceeding as before, the result is

$$\sum_{n=1}^{\infty}d(n)f(n)=\tfrac{1}{4}f(0)+\mathfrak{F}'(1)-2\gamma\mathfrak{F}(1)+$$
$$+\sum_{n=1}^{\infty}d(n)\int_0^{\infty}f(x)[K_0\{4\pi\sqrt{(nx)}\}-\tfrac{1}{2}\pi Y_0\{4\pi\sqrt{(nx)}\}]\,dx.$$

2.10. Examples. (i) Let $f(x)=e^{-x}$, $F_c(x)=\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{1+x^2}$. Then

$$\sqrt{\alpha}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-n\alpha}\right)=\sqrt{\left(\frac{2\beta}{\pi}\right)}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2\beta^2}\right).$$

(ii) Let $f(x)=e^{-\frac{1}{2}x^2}$, $F_c(x)=e^{-\frac{1}{2}x^2}$. Then

$$\sqrt{\alpha}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\frac{1}{2}\alpha^2n^2}\right)=\sqrt{\beta}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\frac{1}{2}\beta^2n^2}\right).$$

(iii) Let $f(x)=e^{-\frac{1}{2}x^2}\cos kx$. Then $F_c(x)=e^{-\frac{1}{2}(k^2+x^2)}\cosh kx$. Hence

$$\sqrt{\alpha}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\frac{1}{2}\alpha^2n^2}\cos k\alpha n\right)=\sqrt{\beta}\,e^{-\frac{1}{2}k^2}\left(\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\frac{1}{2}\beta^2n^2}\cosh k\beta n\right).$$

(iv) The function $f(x) = \dfrac{\sin\sqrt{x}}{1+x}$ satisfies the conditions of Theorem 45, but does not belong to $L(0,\infty)$.

(v) Let $f(x) = x^{-s}\sin^2 x$ $(1 < \sigma < 2)$. Then

$$F_c(x) = \frac{1}{\sqrt{(2\pi)}}\Gamma(1-s)\sin\tfrac{1}{2}s\pi\,(x^{s-1}-\tfrac{1}{2}|x-2|^{s-1}-\tfrac{1}{2}|x+2|^{s-1}) \quad (x > 2),$$

$$F_c(0) = \frac{1}{\sqrt{(2\pi)}}\Gamma(1-s)\sin\tfrac{1}{2}s\pi\,(-2^{s-1}).$$

Hence, taking $\alpha = \frac{1}{2}\pi$, $\beta = 4$,

$$\sqrt{\left(\frac{\pi}{2}\right)}\left\{\left(\frac{\pi}{2}\right)^{-s}+\left(\frac{3\pi}{2}\right)^{-s}+\dots\right\}$$

$$= 2\,\frac{\Gamma(1-s)\sin\frac{1}{2}s\pi}{\sqrt{(2\pi)}}\left\{-2^{s-2}+\sum_{n=1}^{\infty}\left((4n)^{s-1}-\tfrac{1}{2}(4n-2)^{s-1}-\tfrac{1}{2}(4n+2)^{s-1}\right)\right\},$$

or

$$1+\frac{1}{3^s}+\frac{1}{5^s}+\dots = \Gamma(1-s)\sin\tfrac{1}{2}s\pi\,\pi^{s-1}\times$$

$$\times\left\{-\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{(2n)^{1-s}}-\frac{1}{2}\frac{1}{(2n-1)^{1-s}}-\frac{1}{2}\frac{1}{(2n+1)^{1-s}}\right)\right\}.$$

This is the functional equation for $(1-2^{-s})\zeta(s)$.

(vi) Let

$$f(x) = \frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}(1-x^2)^{\nu-\frac{1}{2}} \quad (0 < x < 1), \qquad 0 \quad (x \geqslant 1).$$

Then†

$$F_c(x) = x^{-\nu}J_\nu(x) \quad (x > 0), \qquad F_c(0) = 2^{-\nu}/\Gamma(\nu+1).$$

Hence

$$\frac{1}{2^{\nu+1}\Gamma(\nu+1)}+\sum_{n=1}^{\infty}\frac{J_\nu(n\beta)}{(n\beta)^\nu} = \sqrt{\left(\frac{\alpha}{\beta}\right)}\frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}\left\{\tfrac{1}{2}+\sum_{n\leqslant 1/\alpha}(1-n^2\alpha^2)^{\nu-\frac{1}{2}}\right\},$$

where, in the case $\nu = \frac{1}{2}$, the term $n = 1/\alpha$, if it occurs, is to be halved.

This is a case of Theorem 45 if $\nu \geqslant \frac{1}{2}$. Actually it is easy to see that the same proof applies if $-\frac{1}{2} < \nu < \frac{1}{2}$, provided that α is not the reciprocal of an integer.

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† See (7.1.11).

2.11. Sine transforms. The corresponding theorem is

THEOREM 46. *Let $f(x)$ be integrable over $(0, \delta)$, of bounded variation over (δ, ∞), where $0 < \delta < \frac{1}{2}\pi$, and tend to 0 at infinity. Then*

$$\sqrt{\beta}\{F_s(\beta)-F_s(3\beta)+\dots\}$$
$$= \sqrt{\alpha}[\tfrac{1}{2}\{f(\alpha+0)+f(\alpha-0)\}-\tfrac{1}{2}\{f(3\alpha+0)+f(3\alpha-0)\}+\dots], \qquad (2.11.1)$$

where $\alpha\beta = \frac{1}{2}\pi$.

In this case the right-hand side is necessarily convergent.

Proceding as before, we obtain

$$\sqrt{\beta}[F_s(\beta)-F_s(3\beta)+\dots+(-1)^n F_s\{(2n+1)\beta\}]$$
$$= \frac{(-1)^n\sqrt{\beta}}{\sqrt{(2\pi)}}\int\limits_0^\infty f(t)\frac{\sin(2n+2)\beta t}{\cos\beta t}\,dt$$
$$= \frac{(-1)^n\sqrt{\beta}}{\sqrt{(2\pi)}}\sum_{m=1}^{\infty}\int\limits_{(m-1)\pi/\beta}^{m\pi/\beta} f(t)\frac{\sin(2n+2)\beta t}{\cos\beta t}\,dt$$
$$= \frac{1}{\sqrt{(2\pi\beta)}}\sum_{m=1}^{\infty}(-1)^{m-1}\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f\left\{\frac{(m-\frac{1}{2})\pi+v}{\beta}\right\}\frac{\sin(2n+2)v}{\sin v}\,dv.$$

This differs from the right-hand side of (2.11.1) by

$$\frac{\sqrt{\alpha}}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\int\limits_0^{\frac{1}{2}\pi}\left[f\left\{\frac{(m-\frac{1}{2})\pi+v}{\beta}\right\}-f\left\{\frac{(m-\frac{1}{2})\pi}{\beta}+0\right\}\right]\frac{\sin(2n+2)v}{\sin v}\,dv+$$
$$+(-1)^{m-1}\int\limits_{-\frac{1}{2}\pi}^{0}\left[f\left\{\frac{(m-\frac{1}{2})\pi+v}{\beta}\right\}-f\left\{\frac{(m-\frac{1}{2})\pi}{\beta}-0\right\}\right]\frac{\sin(2n+2)v}{\sin v}\,dv,$$

and the result follows as before.

EXAMPLE. Let $f(x) = x^{-s}$ $(0 < \sigma < 1)$. Then

$$F_s(x) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1-s)\cos\tfrac{1}{2}s\pi\, x^{s-1},$$

and

$$\sqrt{\left(\frac{2\beta}{\pi}\right)}\Gamma(1-s)\cos\tfrac{1}{2}s\pi\{\beta^{s-1}-(3\beta)^{s-1}+\dots\} = \sqrt{\alpha}\{\alpha^{-s}-(3\alpha)^{-s}+\dots\},$$

or $\left(\frac{\pi}{2}\right)^{s-1}\Gamma(1-s)\cos\frac{1}{2}s\pi\, L(1-s) = L(s)$, $\quad L(s) = 1^{s}-\frac{1}{3^s}+\frac{1}{5^s}-\dots.$

2.12. More general conditions. The next theorem is a more general one, in which $f(x)$ is not necessarily of bounded variation.

Theorem 47. *Let $f(x)$ be integrable over any finite interval, let $\alpha\beta = 2\pi$, and*

$$\chi_N(y) = \sum_{n=1}^{N} f(y+n\alpha) \to \chi(y), \tag{2.12.1}$$

and $|\chi_N(y)| \leqslant \phi(y)$, where $\phi(y)$ is $L(-\tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha)$. Then

$$F_c(n\beta) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^{\to\infty} f(x)\cos n\beta x\, dx \tag{2.12.2}$$

exists for every n, and

$$\lim_{\delta\to 0} \sqrt{\beta}\left\{\tfrac{1}{2}F_c(0)+\sum_{n=1}^{\infty} F_c(n\beta)e^{-\delta n\beta}\right\} = \tfrac{1}{2}\sqrt{\alpha}\{f(+0)+\chi(+0)+\chi(-0)\} \tag{2.12.3}$$

provided that the right-hand side exists.[†]

We have

$$\int\limits_{\frac{1}{2}\alpha}^{(N+\frac{1}{2})\alpha} f(x)\cos n\beta x\, dx = \sum_{m=1}^{N} \int\limits_{(m-\frac{1}{2})\alpha}^{(m+\frac{1}{2})\alpha} f(x)\cos n\beta x\, dx$$

$$= \sum_{m=1}^{N} \int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} f(y+m\alpha)\cos n\beta y\, dy = \int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \chi_N(y)\cos n\beta y\, dy.$$

Hence
$$\left|\int\limits_{\frac{1}{2}\alpha}^{(N+\frac{1}{2})\alpha} f(x)\cos n\beta x\, dx\right| \leqslant \int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \phi(y)\, dy.$$

Also, if $(N+\frac{1}{2})\alpha \leqslant X < (N+\frac{3}{2})\alpha$,

$$\left|\int\limits_{(N+\frac{1}{2})\alpha}^{X} f(x)\cos n\beta x\, dx\right| \leqslant \int\limits_{(N+\frac{1}{2})\alpha}^{(N+\frac{3}{2})\alpha} |f(x)|\, dx$$

$$= \int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} |\chi_{N+1}(x)-\chi_N(x)|\, dx \leqslant 2\int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \phi(y)\, dy.$$

Hence
$$\int\limits_0^X f(x)\cos n\beta x\, dx$$

is bounded for all n and X. Also

$$\int\limits_{(N+\frac{1}{2})\alpha}^{(N'+\frac{1}{2})\alpha} f(x)\cos n\beta x\, dx = \int\limits_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \{\chi_{N'}(y)-\chi_N(y)\}\cos n\beta y\, dy$$

tends to 0 as $N \to \infty$, $N' \to \infty$, by Lebesgue's convergence theorem; and similarly

$$\int\limits_{T}^{(N+\frac{1}{2})\alpha} f(x)\cos n\beta x\, dx \to 0, \qquad \int\limits_{(N'+\frac{1}{2})\alpha}^{T'} f(x)\cos n\beta x\, dx \to 0$$

[†] Borgen (1).

if $(N-\frac{1}{2})\alpha < T \leqslant (N+\frac{1}{2})\alpha$, $(N'+\frac{1}{2})\alpha \leqslant T' < (N'+\frac{3}{2})\alpha$, $T \to \infty$, $T' \to \infty$. Hence (2.12.2) converges boundedly for every n.

In (2.8.1) with α and β interchanged, take $f(x) = e^{-\delta x}\cos xy$. Then

$$F_c(t) = \frac{1}{\sqrt{(2\pi)}}\left\{\frac{\delta}{\delta^2+(y-t)^2}+\frac{\delta}{\delta^2+(y+t)^2}\right\},$$

and we obtain

$$\sqrt{\beta}\left\{\tfrac{1}{2}+\sum_{n=1}^{\infty} e^{-\delta n\beta}\cos n\beta y\right\} = \sqrt{\left(\frac{\alpha}{2\pi}\right)}\sum_{n=-\infty}^{\infty}\frac{\delta}{\delta^2+(y-n\alpha)^2}$$
$$= \sqrt{(\tfrac{1}{2}\alpha\pi)}K(y,\delta),$$

say. Hence

$$\sqrt{\beta}\left\{\frac{1}{\sqrt{(2\pi)}}\int_0^Y f(y)\,dy+\sqrt{\left(\frac{2}{\pi}\right)}\sum_{n=1}^{\infty} e^{-\delta n\beta}\int_0^Y \cos n\beta y f(y)\,dy\right\}$$
$$= \sqrt{\alpha}\int_0^Y f(y)K(y,\delta)\,dy.$$

By the bounded convergence of (2.12.2), the left-hand side tends to that of (2.12.3) as $Y \to \infty$. Also since $K(y,\delta)$ is periodic, with period α,

$$\int_0^{\to\infty} f(y)K(y,\delta)\,dy = \left\{\int_0^{\frac{1}{2}\alpha}+\sum_{m=1}^{\infty}\int_{(m-\frac{1}{2})\alpha}^{(m+\frac{1}{2})\alpha}\right\}f(y)K(y,\delta)\,dy$$
$$= \int_0^{\frac{1}{2}\alpha} f(y)K(y,\delta)\,dy+\sum_{m=1}^{\infty}\int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} f(y+m\alpha)K(y,\delta)\,dy$$
$$= \int_0^{\frac{1}{2}\alpha} f(y)K(y,\delta)\,dy+\int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \chi(y)K(y,\delta)\,dy,$$

by the dominated convergence of the series, $K(y,\delta)$ being bounded for a fixed δ. The result now follows from Theorem 17, with $x = 0$, $a = -\frac{1}{2}\alpha$, $b = \frac{1}{2}\alpha$, and $K(0,y,\delta) = K(y,\delta)$.

III

TRANSFORMS OF THE CLASS L^2

3.1. Plancherel's theory of Fourier transforms. The formulae (1.2.1), (1.2.2), connecting a pair of Fourier cosine transforms $f(x)$, $F_c(x)$, express a relation between these functions which is formally symmetrical. But in all the theorems which we have proved so far, the two functions satisfy quite different conditions, so that the symmetry is only formal.

A theory of the reciprocity which is completely symmetrical was first given by Plancherel.[†] It depends, not on ordinary convergence or summability, but on mean convergence.

For complex transforms Plancherel's theorem is

Theorem 48. *Let $f(x)$ be a (real or complex) function of the class $L^2(-\infty,\infty)$, and let*

$$F(x,a) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} f(y)e^{ixy}\, dy. \tag{3.1.1}$$

Then, as $a \to \infty$, $F(x,a)$ converges in mean over $(-\infty,\infty)$ to a function $F(x)$ of $L^2(-\infty,\infty)$; and reciprocally

$$f(x,a) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} F(y)e^{-ixy}\, dy \tag{3.1.2}$$

converges in mean to $f(x)$.

The transforms $f(x)$, $F(x)$ are connected by the formulae

$$F(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} f(y) \frac{e^{ixy}-1}{iy}\, dy, \tag{3.1.3}$$

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} F(y) \frac{e^{-ixy}-1}{-iy}\, dy, \tag{3.1.4}$$

for almost all values of x.

It will be seen from the proof that we might replace $F(x,a)$ by

$$F(x,a,b) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{b} f(y)e^{ixy}\, dy,$$

where $a \to \infty$, $b \to \infty$, in any manner.

[†] Plancherel (1), (2), (3), (4).

At the same time we obtain

THEOREM 49. *If $f(x)$, $F(x)$ and $g(x)$, $G(x)$ are Fourier transforms as in the above theorem,* (2.1.1.), (2.1.2), *and* (2.1.3) *hold.*

For cosine and sine transforms the theory is as follows.

THEOREM 50. *Let $f(x)$ belong to $L^2(0,\infty)$, and let*

$$F_c(x,a) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^a f(y)\cos xy\,dy.$$

Then, as $a \to \infty$, $F_c(x,a)$ converges in mean over $(0,\infty)$ to a function $F_c(x)$ of $L^2(0,\infty)$; and reciprocally

$$f_c(x,a) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^a F_c(y)\cos xy\,dy$$

converges in mean to $f(x)$. We have almost everywhere

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int\limits_0^\infty f(y)\frac{\sin xy}{y}\,dy, \quad f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int\limits_0^\infty F_c(y)\frac{\sin xy}{y}\,dy.$$

THEOREM 51. *The analogue of Theorem 50 for sine transforms holds, with* $\cos xy$ *replaced by* $\sin xy$, *and* $\sin xy$ *by* $1-\cos xy$.

THEOREM 52. (2.1.4), (2.1.5), (2.1.6), *and* (2.1.7) *hold for transforms of L^2.*

The cosine and sine theorems may be obtained by taking $f(x)$ even or odd in the 'complex' theorem.

We shall give several different proofs of these theorems.

3.2. Fourier transforms, first method.† This is suggested by Fourier's formal process (§ 1.1). Let

$$a_\nu = \int\limits_{\nu/\lambda}^{(\nu+1)/\lambda} f(x)\,dx \quad (\nu = 0, \pm 1, \ldots),$$

and
$$\Phi_n(x) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu x/\lambda}.$$

Then if $b > 0$ and $n = [\lambda b]-1$,

$$\lim_{\lambda\to\infty} \Phi_n(x) = \int\limits_{-b}^{b} f(y)e^{ixy}\,dy$$

---

† Titchmarsh (1), (2).

uniformly in any finite interval. For

$$\left|\Phi_n(x)-\int_{-b}^{b} f(y)e^{ixy}\,dy\right| = \left|\sum_{\nu=-n}^{n}\int_{\nu/\lambda}^{(\nu+1)/\lambda} f(y)(e^{i\nu x/\lambda}-e^{ixy})\,dy-\right.$$

$$\left.-\int_{(n+1)/\lambda}^{b} f(y)e^{ixy}\,dy-\int_{-b}^{-n/\lambda} f(y)e^{ixy}\,dy\right|$$

$$\leqslant \frac{x}{\lambda}\int_{-b}^{b}|f(y)|\,dy+\int_{(n+1)/\lambda}^{b}|f(y)|\,dy+\int_{-b}^{-n/\lambda}|f(y)|\,dy \to 0,$$

since $|e^{i\nu x/\lambda}-e^{ixy}| \leqslant x/\lambda$ in each integral.

Also

$$|a_\nu|^2 \leqslant \int_{\nu/\lambda}^{(\nu+1)/\lambda}|f(x)|^2\,dx\int_{\nu/\lambda}^{(\nu+1)/\lambda}dx = \frac{1}{\lambda}\int_{\nu/\lambda}^{(\nu+1)/\lambda}|f(x)|^2\,dx.$$

Hence, if $X \leqslant \pi\lambda$,

$$\int_{-X}^{X}|\Phi_n(x)|^2\,dx \leqslant \int_{-\pi\lambda}^{\pi\lambda}|\Phi_n(x)|^2\,dx = \int_{-\pi\lambda}^{\pi\lambda}\left(\sum_{\nu=-n}^{n}a_\nu e^{i\nu x/\lambda}\sum_{\mu=-n}^{n}\bar{a}_\mu e^{-i\mu x/\lambda}\right)dx$$

$$= 2\pi\lambda\sum_{\nu=-n}^{n}|a_\nu|^2 \leqslant 2\pi\int_{-n/\lambda}^{(n+1)/\lambda}|f(x)|^2\,dx \leqslant 2\pi\int_{-b}^{b}|f(x)|^2\,dx.$$

Keeping X fixed and making $\lambda \to \infty$, it follows that

$$\int_{-X}^{X}\left|\int_{-b}^{b}f(y)e^{ixy}\,dy\right|^2 dx \leqslant 2\pi\int_{-b}^{b}|f(x)|^2\,dx.$$

Making $X \to \infty$,

$$\int_{-\infty}^{\infty}\left|\int_{-b}^{b}f(y)e^{ixy}\,dy\right|^2 dx \leqslant 2\pi\int_{-b}^{b}|f(x)|^2\,dx. \tag{3.2.1}$$

If we take $f(y) = 0$ for $-a < y < a$, this gives

$$\int_{-\infty}^{\infty}|F(x,b)-F(x,a)|^2\,dx \leqslant \left(\int_{-b}^{-a}+\int_{a}^{b}\right)|f(x)|^2\,dx,$$

which tends to 0 as $a \to \infty$, $b \to \infty$. Hence $F(x,a)$ converges in mean, to a function $F(x)$, say, of $L^2(-\infty,\infty)$; and, making $b \to \infty$ in (3.2.1),

$$\int_{-\infty}^{\infty}|F(x)|^2\,dx \leqslant \int_{-\infty}^{\infty}|f(x)|^2\,dx. \tag{3.2.2}$$

A similar argument now shows that $f(x,a)$ converges in mean, to

$\phi(x)$ say. We have to prove that $\phi(x)=f(x)$ almost everywhere, and for this it is sufficient to show that

$$\int\limits_0^\xi \phi(x)\,dx = \int\limits_0^\xi f(x)\,dx$$

for all values of ξ. Now

$$\int\limits_0^\xi \phi(x)\,dx = \lim_{a\to\infty}\int\limits_0^\xi f(x,a)\,dx = \lim_{a\to\infty}\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\xi dx\int\limits_{-a}^{a} F(y)e^{-ixy}\,dy$$

$$= \lim_{a\to\infty}\frac{1}{\sqrt{(2\pi)}}\int\limits_{-a}^{a} F(y)\frac{e^{-i\xi y}-1}{-iy}\,dy = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} F(y)\frac{e^{-i\xi y}-1}{-iy}\,dy.$$

On the other hand, Theorem 22 (1.24.2), with $f(x)=0$ for $|x|>a$, gives

$$\int\limits_0^\xi f(x)\,dx = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{e^{-i\xi u}-1}{-iu}F(u,a)\,du \quad (|\xi|<a).$$

But $$\lim_{a\to\infty}\int\limits_{-\infty}^{\infty}\frac{e^{-i\xi u}-1}{u}F(u,a)\,du = \int\limits_{-\infty}^{\infty}\frac{e^{-i\xi u}-1}{u}F(u)\,du,$$

since $(e^{-i\xi u}-1)/u$ belongs to $L^2(-\infty,\infty)$. The result stated therefore follows.

Incidentally we have proved (3.1.4); since we may now argue similarly with (3.1.2) instead of (3.1.1), (3.1.3) also follows.

Also we may interchange f and F in (3.2.2). Hence in fact

$$\int\limits_{-\infty}^{\infty}|F(x)|^2\,dx = \int\limits_{-\infty}^{\infty}|f(x)|^2\,dx.$$

If $G(x)$ is the transform of $g(x)$ in the same sense, $F+G$ is the transform of $f+g$; hence

$$\int\limits_{-\infty}^{\infty}|F(x)+G(x)|^2\,dx = \int\limits_{-\infty}^{\infty}|f(x)+g(x)|^2\,dx,$$

i.e.

$$\int\limits_{-\infty}^{\infty}\{|F(x)|^2+|G(x)|^2+2\mathbf{R}F(x)\bar{G}(x)\}\,dx$$

$$= \int\limits_{-\infty}^{\infty}\{|f(x)|^2+|g(x)|^2+2\mathbf{R}f(x)\bar{g}(x)\}\,dx.$$

Hence $$\mathbf{R}\int_{-\infty}^{\infty} F(x)\bar{G}(x)\,dx = \mathbf{R}\int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx.$$

Arguing in the same way with $f+ig$, we see that the imaginary parts are also equal. Hence

$$\int_{-\infty}^{\infty} F(x)\bar{G}(x)\,dx = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx.$$

3.3. Fourier transforms, second method.† Let $f(x)$ belong to $L^2(-\infty,\infty)$. Then we can construct a sequence of functions $f_n(x)$, each of which is continuous and of bounded variation over a finite interval, and zero outside this interval, and such that

$$\int_{-\infty}^{\infty} |f(x)-f_n(x)|^2\,dx \to 0.$$

Let $$F_n(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f_n(u)e^{ixu}\,du.$$

Then

$$\int_{-\lambda}^{\lambda} |F_n(x)|^2\,dx = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} dx \int_{-\infty}^{\infty} f_n(u)e^{ixu}\,du \int_{-\infty}^{\infty} \bar{f}_n(v)e^{-ixv}\,dv$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_n(u)\,du \int_{-\infty}^{\infty} \bar{f}_n(v)\,dv \int_{-\lambda}^{\lambda} e^{ix(u-v)}\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_n(u)\,du \int_{-\infty}^{\infty} \bar{f}_n(v)\frac{2\sin\lambda(u-v)}{u-v}\,dv.$$

By the theory of § 1.9, the inner integral tends to $2\pi\bar{f}_n(u)$ uniformly over any finite range, as $\lambda\to\infty$; and hence

$$\int_{-\infty}^{\infty} |F_n(x)|^2\,dx = \int_{-\infty}^{\infty} |f_n(u)|^2\,du.$$

Similarly

$$\int_{-\infty}^{\infty} |F_m(x)-F_n(x)|^2\,dx = \int_{-\infty}^{\infty} |f_m(u)-f_n(u)|^2\,du,$$

---

† Bochner, *Vorlesungen über Fouriersche Integrale*, § 41.

and the right-hand side tends to 0 as m and n tend to infinity. Hence $F_n(x)$ converges in mean, to $F(x)$ say; and

$$\int_{-\infty}^{\infty} |F(x)|^2\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} |F_n(x)|^2\,dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx.$$

This function $F(x)$ is the Fourier transform of $f(x)$. It is of course not yet obvious that it is equivalent to the transform obtained before, or even that it is unique, since the sequence $f_n(x)$ is not unique. However, we have

$$\int_0^{\xi} F_n(x)\,dx = \frac{1}{\sqrt{(2\pi)}} \int_0^{\xi} dx \int_{-\infty}^{\infty} f_n(u)e^{ixu}\,du = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f_n(u)\frac{e^{i\xi u}-1}{iu}\,du$$

(the range of integration being really finite). Making $n \to \infty$,

$$\int_0^{\xi} F(x)\,dx = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\frac{e^{i\xi u}-1}{iu}\,du,$$

since $(e^{i\xi u}-1)/(iu)$ belongs to L^2. Hence

$$F(x) = \frac{1}{\sqrt{(2\pi)}}\frac{d}{dx} \int_{-\infty}^{\infty} f(u)\frac{e^{ixu}-1}{iu}\,du$$

almost everywhere. Hence $F(x)$ is unique (apart from sets of measure zero), and is equivalent to the transform obtained by the first method.

In the first method we deduced the Parseval formula from the reciprocity; in this method we have proved the Parseval formula already, and we deduce the reciprocity from it. As before, the Parseval formula gives

$$\int_{-\infty}^{\infty} F(x)\bar{G}(x)\,dx = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx.$$

Let $g(x) = 1$ $(0 \leqslant x \leqslant \xi)$, $g(x) = 0$ $(x < 0$ or $x > \xi)$. Then

$$G(x) = \frac{1}{\sqrt{(2\pi)}}\frac{d}{dx}\int_0^{\xi} \frac{e^{ixu}-1}{iu}\,du$$

$$= \frac{1}{\sqrt{(2\pi)}}\frac{d}{dx}\int_0^{\xi x} \frac{e^{iu}-1}{iu}\,du = \frac{1}{\sqrt{(2\pi)}}\,\frac{e^{i\xi x}-1}{ix}.$$

Hence
$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} F(x)\frac{e^{-i\xi x}-1}{-ix}\,dx = \int\limits_{0}^{\xi} f(x)\,dx,$$
so that $f(x)$ is the transform of $F(x)$.

Again, let $h(x) = f(x)$ $(-a \leqslant x \leqslant a)$, 0 $(|x| > a)$. Then
$$H(x) = \frac{1}{\sqrt{(2\pi)}}\frac{d}{dx}\int\limits_{-a}^{a} f(u)\frac{e^{ixu}-1}{iu}\,du = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-a}^{a} f(u)e^{ixu}\,du = F(x,a),$$
with the usual notation. Hence the transform of $F(x)-F(x,a)$ is $f(x)-h(x)$, i.e. it is 0 $(|x| \leqslant a)$, $f(x)$ $(|x| > a)$. Hence
$$\int\limits_{-\infty}^{\infty} |F(x)-F(x,a)|^2\,dx = \left(\int\limits_{-\infty}^{-a} + \int\limits_{a}^{\infty}\right)|f(x)|^2\,dx,$$
which tends to 0 as $a \to \infty$. Hence
$$F(x) = \underset{a\to\infty}{\text{l.i.m.}}\ F(x,a).$$

3.4. Fourier transforms, third method.†

Suppose first that $f(x)$ belongs to both L and L^2, and let
$$F(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} f(t)e^{ixt}\,dt.$$
Then
$$\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\delta^2x^2}|F(x)|^2\,dx = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\delta^2x^2}\,dx\int\limits_{-\infty}^{\infty} f(u)e^{ixu}\,du\int\limits_{-\infty}^{\infty} \bar{f}(v)e^{-ixv}\,dv$$
$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(u)\,du\int\limits_{-\infty}^{\infty} \bar{f}(v)\,dv\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\delta^2x^2+ix(u-v)}\,dx$$
$$= \frac{1}{\delta\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} f(u)\,du\int\limits_{-\infty}^{\infty} \bar{f}(v)e^{-\frac{1}{2}(u-v)^2/\delta^2}\,dv, \quad (3.4.1)$$
and by Schwarz's inequality for double integrals
$$\left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(u)\bar{f}(v)e^{-\frac{1}{2}(u-v)^2/\delta^2}\,dudv\right|$$
$$= \left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(u)e^{-\frac{1}{4}(u-v)^2/\delta^2}\bar{f}(v)e^{-\frac{1}{4}(u-v)^2/\delta^2}\,dudv\right|$$

---

† F. Riesz (2).

$$\leqslant \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)|^2 e^{-\frac{1}{2}(u-v)^2/\delta^2}\, dudv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(v)|^2 e^{-\frac{1}{2}(u-v)^2/\delta^2}\, dudv \right\}^{\frac{1}{2}}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)|^2 e^{-\frac{1}{2}(u-v)^2/\delta^2}\, dudv$$

$$= \int_{-\infty}^{\infty} |f(u)|^2\, du \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2/\delta^2}\, dt = \delta\sqrt{(2\pi)} \int_{-\infty}^{\infty} |f(u)|^2\, du.$$

Hence
$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\delta^2 x^2} |F(x)|^2\, dx \leqslant \int_{-\infty}^{\infty} |f(u)|^2\, du,$$

and, making $\delta \rightarrow 0$, it follows that $F(x)$ belongs to $L^2(-\infty,\infty)$.

Also (3.4.1) is equal to

$$\frac{1}{\delta\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\, du \int_{-\infty}^{\infty} \bar{f}(u+t) e^{-\frac{1}{2}t^2/\delta^2}\, dt$$

$$= \frac{1}{\delta\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2/\delta^2}\, dt \int_{-\infty}^{\infty} f(u)\bar{f}(u+t)\, du = \frac{1}{\delta\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2/\delta^2} \psi(t)\, dt$$

say. Since f belongs to L^2, ψ is bounded and continuous. Hence

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = \lim_{\delta \to 0} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\delta^2 x^2} |F(x)|^2\, dx = \lim_{\delta \to 0} \frac{1}{\delta\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2/\delta^2} \psi(t)\, dt$$
$$= \psi(0) = \int_{-\infty}^{\infty} |f(u)|^2\, du,$$

by the theory of Weierstrass's singular integral (§ 1.18).

The existence of F for any f of L^2, and the reciprocity, may now be proved as in the previous method.

3.5. The Hermite polynomials.[†]

The Hermite polynomial of degree n is defined by

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}, \tag{3.5.1}$$

and we write

$$\phi_n(x) = e^{-\frac{1}{2}x^2} H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \tag{3.5.2}$$

The interest of these functions for our theory is that they form an orthogonal sequence, each member of which is, apart from a trivial factor, its own Fourier transform.

[†] See Wiener, *The Fourier Integral*, 51–71.

We have

$$\phi_n''(x) = (-1)^n\left\{(x^2+1)\left(\frac{d}{dx}\right)^n e^{-x^2}+2x\left(\frac{d}{dx}\right)^{n+1} e^{-x^2}+\left(\frac{d}{dx}\right)^{n+2} e^{-x^2}\right\}e^{\frac{1}{2}x^2},$$

and

$$\left(\frac{d}{dx}\right)^{n+2} e^{-x^2} = \left(\frac{d}{dx}\right)^{n+1}(-2xe^{-x^2}) = -2x\left(\frac{d}{dx}\right)^{n+1} e^{-x^2}-2(n+1)\left(\frac{d}{dx}\right)^n e^{-x^2}.$$

Hence

$$\phi_n''(x) = (-1)^n\left\{(x^2+1)\left(\frac{d}{dx}\right)^n e^{-x^2}-2(n+1)\left(\frac{d}{dx}\right)^n e^{-x^2}\right\}e^{\frac{1}{2}x^2}$$

$$= (x^2-2n-1)\phi_n(x). \tag{3.5.3}$$

Thus $y = \phi_n(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}-x^2y = -(2n+1)y. \tag{3.5.4}$$

Putting $y = e^{-\frac{1}{2}x^2}u$, we obtain

$$\frac{d^2u}{dx^2}-2x\frac{du}{dx} = -2nu, \tag{3.5.5}$$

so that $H_n(x)$ is a solution of this equation.

Further, it is the only polynomial solution. For let

$$u = a_0+a_1x+\dots$$

be a solution. Then

$$\sum a_r r(r-1)x^{r-2}-2\sum a_r rx^r = -2n\sum a_r x^r.$$

Hence

$$(r+1)(r+2)a_{r+2} = 2(r-n)a_r,$$

and the general solution is the sum of two series, of which the one in which r has the same parity as n terminates, and the other does not.

The Hermite functions $\phi_n(x)$ form an orthogonal set. For by (3.5.3)

$$\phi_m''(x)\phi_n(x)-\phi_n''(x)\phi_m(x) = 2(n-m)\phi_m(x)\phi_n(x).$$

Hence

$$\int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)\,dx = \frac{1}{2(n-m)}[\phi_m'(x)\phi_n(x)-\phi_n'(x)\phi_m(x)]_{-\infty}^{\infty}$$

$$= 0$$

if $m \neq n$.

3.6. Theorem 53.† *If* $|t| < 1$,

$$\sum_{n=0}^{\infty}\frac{e^{-\frac{1}{2}(x^2+y^2)}}{2^n n!}t^nH_n(x)H_n(y) = \frac{1}{\sqrt{(1-t^2)}}\exp\left\{\frac{x^2-y^2}{2}-\frac{(x-yt)^2}{1-t^2}\right\}. \tag{3.6.1}$$

We have

$$e^{-x^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2+2ixu}\,du,$$

---

† See Watson (3).

Now let $f(x)$ be any function of $L^2(-\infty,\infty)$. Let $f_\nu(x)$ be a continuous function, vanishing outside a finite range, such that

$$\int_{-\infty}^{\infty} |f(x)-f_\nu(x)|^2\,dx < \epsilon,$$

and let
$$a_{n,\nu} = \int_{-\infty}^{\infty} f_\nu(x)\psi_n(x)\,dx.$$

Then

$$\int_{-\infty}^{\infty} \{f(x)-a_{1,\nu}\psi_1(x)-\dots-a_{n,\nu}\psi_n(x)\}^2\,dx$$
$$= \int_{-\infty}^{\infty} \{f(x)\}^2\,dx + \sum_1^n a_{m,\nu}^2 - 2\sum_1^n a_{m,\nu}a_m$$
$$\geqslant \int_{-\infty}^{\infty} \{f(x)\}^2\,dx - \sum_1^n a_m^2 = \int_{-\infty}^{\infty} \{f(x)-a_1\psi_1(x)-\dots-a_n\psi_n(x)\}^2\,dx,$$

and also

$$\leqslant 2\int_{-\infty}^{\infty} \{f(x)-f_\nu(x)\}^2\,dx + 2\int_{-\infty}^{\infty} \{f_\nu(x)-a_{1,\nu}\psi_1(x)-\dots-a_{n,\nu}\psi_n(x)\}^2\,dx$$
$$< 2\epsilon + 2\int_{-\infty}^{\infty} \{f_\nu(x)\}^2\,dx - 2a_{1,\nu}^2 - \dots - 2a_{n,\nu}^2 < 3\epsilon$$

for n sufficiently large. This proves (3.7.2).

Theorem 56. *If $a_1, a_2,\dots$ are given numbers such that $\sum a_n^2$ is convergent, there is a function $f(x)$ of $L^2(-\infty,\infty)$ such that* (3.7.1) *holds.*

This is the Riesz-Fischer theorem for the set $\psi_n(x)$. By Theorem 54

$$\int_{-\infty}^{\infty} \left|\sum_{m=n}^{N} a_m\psi_m(x)\right|^2 dx = \sum_{m=n}^{N} a_m^2,$$

which tends to 0 as n and N tend to infinity. Hence, as $n \to \infty$,

$$\sum_{m=0}^{n} a_m\psi_m(x)$$

converges in mean, to $f(x)$ say; and

$$\int_{-\infty}^{\infty} f(x)\psi_r(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{n} a_m\psi_m(x)\psi_r(x)\,dx$$
$$= a_r.$$

3.8. Theorem 57. *The Fourier transform of $\phi_n(x)$ is $i^n\phi_n(x)$.*

For

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_n(x)e^{ixy}\,dx &= (-1)^n \int_{-\infty}^{\infty} e^{ixy+\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}\,dx \\ &= \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d}{dx}\right)^n e^{ixy+\frac{1}{2}x^2}\,dx \\ &= e^{\frac{1}{2}y^2}\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d}{dx}\right)^n e^{\frac{1}{2}(x+iy)^2}\,dx \\ &= (-i)^n e^{\frac{1}{2}y^2}\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d}{dy}\right)^n e^{\frac{1}{2}(x+iy)^2}\,dx \\ &= (-i)^n e^{\frac{1}{2}y^2}\left(\frac{d}{dy}\right)^n \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+ixy-\frac{1}{2}y^2}\,dx \\ &= (-i)^n e^{\frac{1}{2}y^2}\left(\frac{d}{dy}\right)^n \sqrt{(2\pi)}e^{-y^2} \\ &= i^n\sqrt{(2\pi)}\phi_n(y).\end{aligned}$$

Alternatively, let

$$\Phi_n(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \phi_n(x)e^{ixy}\,dx.$$

Then $$\Phi_n''(y) = -\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} x^2\phi_n(x)e^{ixy}\,dx.$$

Also, integrating by parts twice,

$$\Phi_n(y) = -\frac{1}{y^2\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \phi_n''(x)e^{ixy}\,dx.$$

Hence

$$\begin{aligned}&\Phi_n''(y)-y^2\Phi_n(y)+(2n+1)\Phi_n(y) \\ &\quad = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \{\phi_n''(x)-x^2\phi_n(x)+(2n+1)\phi_n(x)\}e^{ixy}\,dx \\ &\quad = 0.\end{aligned}$$

Thus $\Phi_n(x)$ satisfies the same differential equation as $\phi_n(x)$; and it is easily seen that, if $e^{\frac{1}{2}x^2}\phi_n(x)$ is a polynomial, so is $e^{\frac{1}{2}x^2}\Phi_n(x)$.

Hence
$$\Phi_n(x) = c_n\,\phi_n(x).$$

Now
$$e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2+ixy}\,dy,$$
$$\left(\frac{d}{dx}\right)^n e^{-\frac{1}{2}x^2} = \frac{i^n}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}y^n e^{ixy}\,dy,$$

and also
$$\left(\frac{d}{dx}\right)^n e^{-\frac{1}{2}x^2} = \{(-1)^n x^n + \ldots\}e^{-\frac{1}{2}x^2}.$$

Hence
$$\frac{i^n}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}y^n e^{ixy}\,dy = \{(-1)^n x^n + \ldots\}e^{-\frac{1}{2}x^2},$$

and hence
$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} \phi_n(y)e^{ixy}\,dy = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} (2^n y^n + \ldots)e^{-\frac{1}{2}y^2}e^{ixy}\,dy$$
$$= i^n(2^n x^n + \ldots)e^{-\frac{1}{2}x^2}.$$

Hence
$$c_n = i^n.$$

3.9. Fourier transforms, fourth method. Take, for example, an even function $f(x)$ of $L^2(0,\infty)$, and let
$$a_n = \int\limits_{-\infty}^{\infty} f(x)\psi_n(x)\,dx.$$

Then
$$a_1 = a_3 = \ldots = 0,$$
$$a_{2n} = 2\int\limits_0^{\infty} f(x)\psi_{2n}(x)\,dx,$$

and
$$\sum_{n=0}^{\infty} a_{2n}^2 = 2\int\limits_0^{\infty} \{f(x)\}^2\,dx.$$

By Theorem 56 there is an even function $g(x)$ such that
$$(-1)^n a_{2n} = 2\int\limits_0^{\infty} g(x)\psi_{2n}(x)\,dx \quad (n = 0, 1, \ldots);$$

and
$$\int\limits_0^{\infty} \{g(x)\}^2\,dx = \sum_{n=0}^{\infty} a_{2n}^2 = \int\limits_0^{\infty} \{f(x)\}^2\,dx.$$

The relation between f and g is plainly reciprocal.

We now identify $g(x)$ with the Fourier cosine transform of $f(x)$ previously obtained.

We have

$$\int_0^\infty \frac{\sin xy}{y}\psi_{2n}(y)\,dy = (-1)^n\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{\sin xy}{y}\,dy\int_0^\infty \psi_{2n}(t)\cos yt\,dt$$

$$= (-1)^n\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \psi_{2n}(t)\,dt\int_0^\infty \frac{\sin xy\cos yt}{y}\,dy = (-1)^n\sqrt{\left(\frac{\pi}{2}\right)}\int_0^x \psi_{2n}(t)\,dt,$$

since $\psi_{2n}(t)$ belongs to $L(0,\infty)$ and the y-integral converges boundedly. Hence

$$\int_0^\infty g(y)\frac{\sin xy}{y}\,dy = \lim_{N\to\infty}\int_0^\infty \frac{\sin xy}{y}\sum_{n=0}^{N}(-1)^n a_{2n}\psi_{2n}(y)\,dy$$

$$= \lim_{N\to\infty}\sqrt{\left(\frac{\pi}{2}\right)}\sum_{n=0}^{N} a_{2n}\int_0^x \psi_{2n}(t)\,dt = \sqrt{\left(\frac{\pi}{2}\right)}\int_0^x f(t)\,dt,$$

so that f and g are Fourier transforms in the ordinary sense.

Similarly, by taking $f(x)$ odd, we obtain the theory of sine transforms.

3.10. Convergence and summability.

We can now prove theorems for L^2 functions corresponding to Theorems 3 and 14.

THEOREM 58. *If $f(t)$ belongs to $L^2(-\infty,\infty)$, and is of bounded variation in the neighbourhood of $t = x$, then*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} F(u)e^{-ixu}\,du.$$

The transform of $G(u) = e^{-ixu}$ ($|u| < \lambda$), 0 ($|u| > \lambda$), is

$$g(v) = \frac{1}{\sqrt{(2\pi)}}\int_{-\lambda}^{\lambda} e^{-ixu-iuv}\,du = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\sin\lambda(x+v)}{x+v}.$$

Hence, by Parseval's formula,

$$\int_{-\lambda}^{\lambda} F(u)e^{-ixu}\,du = \sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty} f(v)\frac{\sin\lambda(x-v)}{x-v}\,dv.$$

The result now follows from the theory of Fourier's single integral (Theorem 12, case i (a)).

Theorem 59.[†] *If $f(t)$ belongs to $L^2(-\infty,\infty)$, then*

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)e^{-ixu}\,du$$

wherever
$$\int_0^h |f(x+t)+f(x-t)-2f(x)|\,dt = o(h)$$

as $h \to 0$, and so for almost all values of x; and Fourier's repeated integral for $f(x)$ holds almost everywhere, if both integrals are taken in the $(C,1)$ sense.

The transform of $G(u) = \left(1-\frac{|u|}{\lambda}\right)e^{-ixu}$ $(|u| < \lambda)$, 0 $(|u| > \lambda)$, is

$$g(v) = \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) e^{-ixu-iuv}\,du = \sqrt{\left(\frac{2}{\pi}\right)}\, 2\,\frac{\sin^2\frac{1}{2}\lambda(x+v)}{\lambda(x+v)^2}.$$

Hence by Parseval's formula

$$\int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right) F(u)e^{-ixu}\,du = \sqrt{\left(\frac{2}{\pi}\right)} \int_{-\infty}^{\infty} f(v)\, 2\,\frac{\sin^2\frac{1}{2}\lambda(x-v)}{\lambda(x-v)^2}\,dv,$$

and, as in § 1.16, the result follows from Theorem 13.

The result also holds with f and F interchanged, and this gives the second part of the theorem.

We also deduce

Theorem 60. *If*

$$f(x) = \underset{a\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} \psi(t)e^{-ixt}\,dt,$$

where ψ belongs to L^2, and also

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \chi(t)e^{-ixt}\,dt,$$

where χ belongs to L, then $\psi \equiv \chi$.

For by the above theorem

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} \left(1-\frac{|t|}{\lambda}\right) f(t)e^{ixt}\,dt$$

is equal to $\psi(x)$ almost everywhere; and by Theorem 14 it is equal to $\chi(x)$ almost everywhere.

[†] Plancherel (3).

3.11. Convergence almost everywhere. If $f(t)$ belongs to L^2, the integral

$$\int_{-\infty}^{\infty} f(t)e^{ixt}\,dt$$

converges in mean; it is also summable $(C, 1)$ almost everywhere, by Theorem 59, since in this theorem f and F are interchangeable.

It is not known whether the integral necessarily converges in the ordinary sense almost everywhere. As in Theorem 58 it would be easy to make it converge almost everywhere by imposing extra conditions on $F(x)$. The object of the next sections is to state simple additional conditions on $f(x)$ itself which make the integral converge almost everywhere.

Theorem 61.† *If $f(t)$ belongs to $L^2(-\infty,\infty)$, then*

$$\int_{-\lambda}^{\lambda} f(t)e^{ixt}\,dt = o(\log\lambda) \tag{3.11.1}$$

wherever

$$\chi(h) = \int_{0}^{h} |F(x+y)+F(x-y)-2F(x)|\,dy = o(h)$$

as $h \to 0$; and so almost everywhere.

As in the proof of Theorem 58

$$\int_{-\lambda}^{\lambda} f(t)e^{ixt}\,dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_{-\infty}^{\infty} F(x+y)\frac{\sin\lambda y}{y}\,dy.$$

Now

$$\left|\int_{1}^{\infty} F(x+y)\frac{\sin\lambda y}{y}\,dy\right|^2 \leqslant \int_{1}^{\infty} |F(x+y)|^2\,dy \int_{1}^{\infty} \frac{dy}{y^2} \leqslant \int_{-\infty}^{\infty} |F(t)|^2\,dt$$

and similarly for the integral over $(-\infty, -1)$. Also

$$\int_{-1}^{1} F(x+y)\frac{\sin\lambda y}{y}\,dy = \int_{0}^{1} \{F(x+y)+F(x-y)\}\frac{\sin\lambda y}{y}\,dy$$

$$= \int_{0}^{1} \{F(x+y)+F(x-y)-2F(x)\}\frac{\sin\lambda y}{y}\,dy + O(1)$$

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† Plancherel (3).

if $F(x)$ is finite. The modulus of this integral does not exceed

$$\lambda\int_0^{1/\lambda}|F(x+y)+F(x-y)-2F(x)|\,dy+\int_{1/\lambda}^{1}|F(x+y)+F(x-y)-2F(x)|\frac{dy}{y}.$$

If $\chi(h)=o(h)$, the first term is $o(1)$, and the second is

$$\int_{1/\lambda}^{1}\chi'(y)\frac{dy}{y}=\left[\frac{\chi(y)}{y}\right]_{1/\lambda}^{1}+\int_{1/\lambda}^{1}\frac{\chi(y)}{y^2}\,dy=O(1)+o\int_{1/\lambda}^{1}\frac{dy}{y}=o(\log\lambda).$$

This proves the theorem.

Theorem 62.† *If $f(t)$ and also $f(t)\log(|t|+2)$ belong to $L^2(-\infty,\infty)$, then*

$$\lim_{\lambda\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-\lambda}^{\lambda}f(t)e^{ixt}\,dt=F(x)\tag{3.11.2}$$

almost everywhere.

We have

$$\int_{-\lambda}^{\lambda}f(t)e^{ixt}\,dt=\int_{-\lambda}^{\lambda}\left(1-\frac{|t|}{\lambda}\right)f(t)e^{ixt}\,dt+\frac{1}{\lambda}\int_{-\lambda}^{\lambda}|t|f(t)e^{ixt}\,dt.$$

The first term tends to $\sqrt{(2\pi)}F(x)$ almost everywhere, by Theorem 59. Hence it is sufficient to prove that

$$\int_{-\lambda}^{\lambda}|t|f(t)e^{ixt}\,dt=o(\lambda)$$

almost everywhere. By Theorem 61

$$\phi(\lambda)=\int_0^{\lambda}f(t)\log(t+2)e^{ixt}\,dt=o(\log\lambda)$$

almost everywhere. If x is a point where this is true, then

$$\begin{aligned}\int_0^{\lambda}tf(t)e^{ixt}\,dt&=\int_0^{\lambda}\frac{t\phi'(t)}{\log(t+2)}\,dt\\&=\left[\frac{t\phi(t)}{\log(t+2)}\right]_0^{\lambda}-\int_0^{\lambda}\left\{\frac{1}{\log(t+2)}-\frac{t}{(t+2)\log^2(t+2)}\right\}\phi(t)\,dt\\&=\frac{\lambda\,o(\log\lambda)}{\log(\lambda+2)}+\int_0^{\lambda}o(1)\,dt=o(\lambda),\end{aligned}$$

and the theorem follows.

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† Plancherel (3).

3.12. In spite of the satisfactory appearance of the above analysis, it is possible to improve on the result.

THEOREM 63. *If $f(t)$ and also $f(t)\sqrt{\log(|t|+2)}$ belong to $L^2(-\infty,\infty)$, then* (3.11.2) *holds almost everywhere.*†

Let
$$\Phi(x) = \int\limits_{-\lambda(x)}^{\lambda(x)} f(t)e^{ixt}\,dt = \int\limits_{-\infty}^{\infty} f(t)\omega(x,t)e^{ixt}\,dt,$$
where $\lambda(x)$ is any function of x such that $\lambda(x) \leqslant a$, and where $\omega(x,t) = 1$ $(|t| \leqslant \lambda(x))$, 0 $(|t| > \lambda(x))$, so that $\omega(x,t) = 0$ for $|t| > a$ and every x. Then
$$\int\limits_0^{\xi} \Phi(x)\,dx = \int\limits_0^{\xi} dx \int\limits_{-\infty}^{\infty} f(t)\omega(x,t)e^{ixt}\,dt = \int\limits_{-\infty}^{\infty} f(t)\sqrt{\{\log(|t|+2)\}}\chi(t)\,dt,$$
where
$$\chi(t) = \frac{1}{\sqrt{\{\log(|t|+2)\}}} \int\limits_0^{\xi} \omega(x,t)e^{ixt}\,dx,$$
(the t-range being really finite). Hence
$$\left|\int\limits_0^{\xi} \Phi(x)\,dx\right|^2 \leqslant \int\limits_{-\infty}^{\infty} |f(t)|^2\log(|t|+2)\,dt \int\limits_{-\infty}^{\infty} |\chi(t)|^2\,dt.$$
Now
$$\int\limits_{-\infty}^{\infty} |\chi(t)|^2\,dt = \int\limits_{-\infty}^{\infty} \frac{dt}{\log(|t|+2)} \int\limits_0^{\xi} \omega(x,t)e^{ixt}\,dx \int\limits_0^{\xi} \omega(y,t)e^{-iyt}\,dy$$
$$= \int\limits_0^{\xi} dx \int\limits_0^{\xi} dy \int\limits_{-\infty}^{\infty} \frac{\omega(x,t)\omega(y,t)}{\log(|t|+2)} e^{i(x-y)t}\,dt$$
$$= \int\limits_0^{\xi} dx \int\limits_0^{\xi} dy \int\limits_{-\lambda(x,y)}^{-\lambda(x,y)} \frac{e^{i(x-y)t}}{\log(|t|+2)}\,dt,$$
where $\lambda(x,y) = \min\{\lambda(x),\lambda(y)\}$. Writing $\lambda(x,y) = \lambda$, we have, on integrating by parts twice,
$$\int\limits_0^{\lambda} \frac{\cos(x-y)t}{\log(t+2)}\,dt = \frac{\sin(x-y)\lambda}{(x-y)\log(\lambda+2)} + \frac{1-\cos(x-y)\lambda}{(x-y)^2(\lambda+2)\log^2(\lambda+2)} +$$
$$+\frac{1}{(x-y)^2}\int\limits_0^{\lambda} \{1-\cos(x-y)t\}\frac{\log(t+2)+2}{(t+2)^2\log^3(t+2)}\,dt$$
$$= J_1+J_2+J_3,$$

---

† The theorem for series is given by Plessner (2).

say. We now observe that *if* $F(x,y,t) = F(y,x,t)$, *then*

$$\left|\int_0^\xi \int_0^\xi F\{x,y,\lambda(x,y)\}\,dxdy\right| \leqslant 2\int_0^\xi dx \int_0^\xi |F\{x,y,\lambda(x)\}|\,dy.$$

For let Q be the square $0 \leqslant x \leqslant \xi$, $0 \leqslant y \leqslant \xi$, and let

$$Q_1 = Q\{\lambda(x) \leqslant \lambda(y)\}, \qquad Q_2 = Q\{\lambda(x) > \lambda(y)\}.$$

Then in Q_1, $\lambda(x,y) = \lambda(x)$, and in Q_2, $\lambda(x,y) = \lambda(y)$. Hence

$$\begin{aligned}\left|\iint_Q F\{x,y,\lambda(x,y)\}\,dxdy\right| & \\ &\leqslant \iint_{Q_1} |F\{x,y,\lambda(x)\}|\,dxdy + \iint_{Q_2} |F\{x,y,\lambda(y)\}|\,dxdy \\ &\leqslant 2\int_0^\xi dx \int_0^\xi |F\{x,y,\lambda(x)\}|\,dy\end{aligned}$$

by symmetry.

It follows that

$$\left|\int_0^\xi \int_0^\xi J_1\,dxdy\right| \leqslant 2\int_0^\xi dx \int_0^\xi \left|\frac{\sin\{(x-y)\lambda(x)\}}{(x-y)\log\{\lambda(x)+2\}}\right|\,dy.$$

Now if $0 \leqslant x \leqslant \xi$,

$$\begin{aligned}\int_0^\xi \left|\frac{\sin\{(x-y)\lambda(x)\}}{x-y}\right|\,dy \leqslant 2\int_0^{\xi\lambda(x)} \left|\frac{\sin u}{u}\right|\,du \leqslant 2\left\{\int_1^{\xi\{\lambda(x)+2\}} \frac{du}{u}+1\right\} \\ = 2[1+\log\xi+\log\{\lambda(x)+2\}].\end{aligned}$$

Hence

$$\left|\int_0^\xi \int_0^\xi J_1\,dxdy\right| \leqslant 4\int_0^\xi \frac{1+\log\xi+\log\{\lambda(x)+2\}}{\log\{\lambda(x)+2\}}\,dx < K(\xi).$$

Similarly

$$\begin{aligned}\left|\int_0^\xi \int_0^\xi J_2\,dxdy\right| &\leqslant 2\int_0^\xi \frac{dx}{\{\lambda(x)+2\}\log^2\{\lambda(x)+2\}} \int_0^\xi \frac{1-\cos\{(x-y)\lambda(x)\}}{(x-y)^2}\,dy \\ &< A\int_0^\xi \frac{\lambda(x)}{\{\lambda(x)+2\}\log^2\{\lambda(x)+2\}}\,dx < K(\xi).\end{aligned}$$

Also

$$\left|\int_0^\xi \int_0^\xi J_3\,dxdy\right| \leqslant \int_0^\infty \frac{\log(t+2)+2}{(t+2)^2\log^3(t+2)}\,dt \int_0^\xi dx \int_0^\xi \frac{1-\cos(x-y)t}{(x-y)^2}\,dy$$

$$< K(\xi) \int_0^\infty \frac{t\{\log(t+2)+2\}}{(t+2)^2\log^3(t+2)}\,dt = K(\xi).$$

Hence, for every $\lambda(x)$,

$$\left|\int_0^\xi \Phi(x)\,dx\right|^2 < K(\xi) \int_{-\infty}^\infty \{f(t)\}^2\log(|t|+2)\,dt.$$

Let $$\phi(x,T,T') = \max_{T\leqslant\lambda(x)\leqslant T'} \int_T^{\lambda(x)} f(t)\cos xt\,dt.$$

Then $\phi(x,T,T')$ is the difference between the real parts of two integrals of type Φ, in which $f(t) = 0$ for $t < T$ and $t > T'$. Hence

$$\left|\int_0^\xi \phi(x,T,T')\,dx\right|^2$$

$$< K(\xi) \int_T^{T'} \{f(t)\}^2\log(t+2)\,dt < K(\xi) \int_T^\infty \{f(t)\}^2\log(t+2)\,dt.$$

As $T' \to \infty$,

$$\phi(x,T,T') \to \phi(x,T) = \max_{T\leqslant\lambda(x)} \int_T^{\lambda(x)} f(t)\cos xt\,dt,$$

and $\phi(x,T,T') \geqslant 0$, since $\int_T^{\lambda(x)} f(t)\cos xt\,dt = 0$ if $\lambda(x) = T$. Hence†

$$\left|\int_0^\xi \phi(x,T)\,dx\right|^2 < K(\xi) \int_T^\infty |f(t)|^2\log(t+2)\,dt.$$

It is then clear that, given ϵ, we can choose a sequence T_1, T_2,... such that $\phi(x,T_n) \to 0$ except when x lies in a part of the interval $(0,\xi)$ of measure less than ϵ. A similar argument applies to the function $\psi(x,T)$ defined with 'min' instead of 'max'. Since

$$\left|\int_u^{u'} f(t)\cos xt\,dt\right| = \left|\int_{T_n}^{u'} - \int_{T_n}^{u}\right|$$

$$\leqslant \phi(x,T_n)-\psi(x,T_n)$$

if $T_n \leqslant u < u'$, it follows that $\int f(t)\cos xt\,dt$ converges for $0 \leqslant x \leqslant \xi$ except for a set of measure less than ϵ.

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† By Fatou's theorem; Titchmarsh, *Theory of Functions*, § 10.81.

Hence
$$\int\limits_0^{\to\infty} f(t)\cos xt\,dt$$
exists for almost all x. Similarly, the sine integral converges for almost all x. Hence the limit (3.11.2) exists almost everywhere; since the limit and limit in mean of a sequence are equal almost everywhere, by Theorem 48 its value is almost everywhere $F(x)$.

3.13. Theorems on resultants.

Theorem 64. *If $f(x)$ and $g(x)$ belong to $L^2(-\infty,\infty)$, then* (2.1.9) *are transforms in the sense that* (2.1.8) *holds for all values of x.*

For a fixed t, the transform of $f(u+t)$ is
$$\operatorname*{l.i.m.}_{a\to\infty}\frac{1}{\sqrt{(2\pi)}}\int\limits_{-a}^{a} f(u+t)e^{ixu}\,du = \operatorname*{l.i.m.}_{a\to\infty}\frac{e^{-ixt}}{\sqrt{(2\pi)}}\int\limits_{-a+t}^{a+t} f(v)e^{ixv}\,dv$$
$$= F(x)e^{-ixt}.$$
The result therefore follows from Parseval's formula, Theorem 49.

Theorem 65. *If $f(x)$ belongs to $L^2(-\infty,\infty)$, and $g(x)$ to $L(-\infty,\infty)$, then* (2.1.9) *are transforms of the class L^2.*

Since F belongs to L^2 and G is bounded, FG belongs to L^2. The integral of its transform is
$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} F(u)G(u)\frac{e^{-ixu}-1}{-iu}\,du. \tag{3.13.1}$$
Now the transform of $G(u)(e^{-ixu}-1)/(-iu)$ is
$$\operatorname*{l.i.m.}_{a\to\infty}\frac{1}{\sqrt{(2\pi)}}\int\limits_{-a}^{a} G(u)\frac{e^{-ixu}-1}{-iu}e^{-ivu}\,du$$
$$= \operatorname*{l.i.m.}_{a\to\infty}\frac{1}{2\pi}\int\limits_{-a}^{a}\frac{e^{-ixu}-1}{-iu}e^{-ivu}\,du\int\limits_{-\infty}^{\infty} g(\xi)e^{i\xi u}\,d\xi$$
$$= \operatorname*{l.i.m.}_{a\to\infty}\frac{1}{\pi}\int\limits_{-\infty}^{\infty} g(\xi)\,d\xi\int\limits_{0}^{a}\frac{\sin(x+y-\xi)u-\sin(y-\xi)u}{u}\,du$$
$$= \int\limits_{y}^{x+y} g(\xi)\,d\xi \quad (x>0),$$

the ordinary limit existing by dominated convergence. Hence (3.13.1) is equal to

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy \int_{-y}^{x-y} g(\xi)\,d\xi$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy \int_{0}^{x} g(u-y)\,du$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{0}^{x} du \int_{-\infty}^{\infty} f(y)g(u-y)\,dy,$$

this inversion being justified by absolute convergence. The theorem now follows on differentiating with respect to x.

The direct proof that, if f is L^2 and g is L, then

$$h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

is L^2, follows from the inequalities

$$|h(x)|^2 \leqslant \int_{-\infty}^{\infty} |f(y)|^2|g(x-y)|dy \int_{-\infty}^{\infty} |g(x-y)|\,dy$$

$$= \int_{-\infty}^{\infty} |f(y)|^2|g(x-y)|\,dy \int_{-\infty}^{\infty} |g(u)|\,du,$$

$$\int_{-\infty}^{\infty} |h(x)|^2\,dx \leqslant \int_{-\infty}^{\infty} |g(u)|\,du \int_{-\infty}^{\infty} |f(y)|^2\,dy \int_{-\infty}^{\infty} |g(x-y)|\,dx$$

$$= \int_{-\infty}^{\infty} |f(y)|^2\,dy \left(\int_{-\infty}^{\infty} |g(u)|\,du\right)^2.$$

THEOREM 66. *If $f(x)$ is positive and $L(-\infty,\infty)$, and $F(x)$ is its transform, then $F(x)$ is of the form*

$$F(x) = \int_{-\infty}^{\infty} \phi(t)\phi(x-t)\,dt, \tag{3.13.2}$$

where ϕ is $L^2(-\infty,\infty)$; and conversely, if $F(x)$ is of this form, then it is the transform of a function $f(x)$ which is positive and L.

If $f(x)$ is positive and L, $\sqrt{f(x)}$ is L^2; let $G(x)$ be its transform. Then G is L^2, and by Theorem 64

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(t)G(x-t)\,dt$$

is the transform of $\sqrt{f}.\sqrt{f} = f$.

Conversely, if (3.13.2) holds, then $f/\sqrt{(2\pi)}$ is the square of the transform of ϕ, and so is positive and L.

THEOREM 67. *If f is L, then $F = \int_{-\infty}^{\infty}\phi(t)\psi(x-t)\,dt$, where ϕ and ψ are L^2; and conversely.*

For $f = \sqrt{|f|}\times\sqrt{|f|}\operatorname{sgn} f$.

3.14. Special theorems. THEOREM 68. *If both $f(x)$ and $f'(x)$ belong to L^2, then both $F(x)$ and $xF(x)$ belong to L^2; and conversely.*

We have $$\{f(x)\}^2-\{f(0)\}^2 = 2\int_0^x f(t)f'(t)\,dt,$$

which tends to a limit as $x\to\infty$. Since $\{f(x)\}^2$ belongs to L, it cannot tend to a limit other than 0, and so tends to 0 as $x\to\infty$. Now

$$\int_{-a}^{a} f'(u)e^{ixu}\,du = [f(u)e^{ixu}]_{-a}^{a}-ix\int_{-a}^{a} f(u)e^{ixu}\,du.$$

As $a\to\infty$, the left-hand side converges in mean square, to $\sqrt{(2\pi)}\Phi(x)$ say. The first term on the right tends to 0, uniformly in x. The second term on the right converges in mean square to $-\sqrt{(2\pi)}ixF(x)$, at any rate over a finite interval. Hence

$$\Phi(x) = -ixF(x),$$

and since $\Phi(x)$ belongs to L^2 the result follows.

Conversely, if $xF(x)$ is L^2, let $\phi(x)$ be its transform. Then

$$\int_0^x \phi(u)\,du = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} uF(u)\frac{e^{-ixu}-1}{-iu}\,du$$

$$= \frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(u)e^{-ixu}\,du-C$$

say, $F(u)$ being $L(-\infty,\infty)$, since F and xF are L^2. But the first term on the right is $if(x)$ almost everywhere, and we may take it as the definition of $if(x)$ everywhere. Then

$$\int_0^x \phi(u)\,du = if(x)-C$$

everywhere, and the result follows.

The result can obviously be extended to any number of derivatives.

3.15. THEOREM 69. *If $f(x)$, $F_c(x)$ are cosine transforms of L^2, so are*

$$\frac{1}{x}\int_0^x f(t)\,dt, \qquad \int_x^\infty \frac{F_c(t)}{t}\,dt;$$

and similarly for sine transforms.

That the second pair of functions belong to L^2 is a theorem of Hardy. (See Titchmarsh, *Theory of Functions*, p. 396.)

To prove that they are transforms, we have

$$\frac{1}{x}\int_0^x f(t)\,dt = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F_c(y)\frac{\sin xy}{xy}\,dy.$$

The cosine transform of this is

$$\frac{2}{\pi}\frac{d}{du}\int_0^\infty \frac{\sin xu}{x}\,dx\int_0^\infty F_c(y)\frac{\sin xy}{xy}\,dy$$

$$= \frac{2}{\pi}\frac{d}{du}\int_0^\infty \frac{F_c(y)}{y}\,dy\int_0^\infty \frac{\sin xu\,\sin xy}{x^2}\,dx$$

$$= \frac{d}{du}\int_0^\infty \frac{F_c(y)}{y}\min(u,y)\,dy$$

$$= \frac{d}{du}\left\{\int_0^u F_c(y)\,dy + u\int_u^\infty \frac{F_c(y)}{y}\,dy\right\} = \int_u^\infty \frac{F_c(y)}{y}\,dy,$$

almost everywhere.

The inversion is justified by absolute convergence. In fact, if $y < u$,

$$\int_0^\infty \left|\frac{\sin xu\,\sin xy}{x^2}\right|\,dx \leqslant \int_0^{1/u} uy\,dx + \int_{1/u}^{1/y} \frac{y}{x}\,dx + \int_{1/y}^\infty \frac{dx}{x^2}$$

$$= y+y\log\frac{u}{y}+y = \left(2+\log\frac{u}{y}\right)y,$$

and, similarly, if $u < y$; and the integral

$$\int_0^u |F_c(y)|\left(2+\log\frac{u}{y}\right)\,dy+u\int_u^\infty \frac{|F_c(y)|}{y}\left(2+\log\frac{y}{u}\right)\,dy$$

is convergent if F_c belongs to L^2.

3.16. Another case of Parseval's formula.

THEOREM 70. *If f is L, and g is L^2 and bounded, then*

$$\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)F(x)G(x)\,dx=\int_{-\infty}^{\infty}f(x)g(-x)\,dx.$$

We have

$$\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)F(x)G(x)\,dx=\frac{1}{\sqrt{(2\pi)}}\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)G(x)\,dx\int_{-\infty}^{\infty}f(t)e^{ixt}\,dt$$

$$=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(t)\,dt\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)G(x)e^{ixt}\,dx,$$

inverting by uniform convergence. As in the proof of Theorem 59, the inner integral is equal to

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty}g(u)\frac{\sin^2\frac{1}{2}\lambda(x+u)}{\lambda(x+u)^2}\,du.$$

This is bounded if $g(u)$ is bounded, and tends almost everywhere to $\sqrt{(2\pi)}g(-x)$. The result therefore follows by dominated convergence.

3.17. Mellin transforms. We shall say that $f(x)$ belongs to $\mathfrak{L}^2$ if

$$\int_0^\infty |f(x)|^2\frac{dx}{x}<\infty.$$

THEOREM 71. *Let $x^kf(x)$ belong to $\mathfrak{L}^2$. Then*

$$\mathfrak{F}(s,a)=\int_{1/a}^{a}f(x)x^{s-1}\,dx\quad(\mathbf{R}(s)=k)$$

converges in mean square over $(k-i\infty,k+i\infty)$, to $\mathfrak{F}(s)$ say;

$$f(x,a)=\frac{1}{2\pi i}\int_{k-ia}^{k+ia}\mathfrak{F}(s)x^{-s}\,ds$$

converges in mean to $f(x)$, in the sense that

$$\lim_{a\to\infty}\int_0^\infty |f(x)-f(x,a)|^2x^{2k-1}\,dx = 0;$$

and
$$\int_0^\infty |f(x)|^2x^{2k-1}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}(k+it)|^2\,dt.$$

This follows from Plancherel's theorem by the usual transformation.

THEOREM 72. *Let $x^kf(x)$ and $x^{1-k}g(x)$ belong to $\mathfrak{L}^2$. Then*

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\mathfrak{G}(1-s)\,ds.$$

This is the corresponding transformation of Theorem 49.

THEOREM 73. *Let $x^kf(x)$ and $x^{\sigma-k}g(x)$ belong to $\mathfrak{L}^2$. Then*

$$f(x)g(x), \qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(w)\mathfrak{G}(s-w)\,dw$$

are Mellin transforms in the sense that

$$\int_0^\infty f(x)g(x)x^{s-1}\,dx = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(w)\mathfrak{G}(s-u)\,dw$$

for all values of t.

This is obtained by replacing $g(x)$ by $g(x)x^{s-1}$ in the previous theorem.

IV

TRANSFORMS OF OTHER L-CLASSES

4.1. Transforms of functions of L^p. PLANCHEREL's theorem can be extended from the exponent 2 to a general exponent p. Throughout the chapter we write $p' = p/(p-1)$, and similarly for other letters.

THEOREM 74.† *Let $f(x)$ belong to $L^p(-\infty,\infty)$, where $1 < p \leqslant 2$. Then, as $a \to \infty$,*

$$F(x,a) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} f(t)e^{ixt}\,dt \tag{4.1.1}$$

converges in mean with exponent p'. The mean limit $F(x)$, called the transform of $f(x)$, satisfies

$$\int_{-\infty}^{\infty} |F(x)|^{p'}\,dx \leqslant \frac{1}{(2\pi)^{\frac{1}{2}p'-1}} \left\{ \int_{-\infty}^{\infty} |f(x)|^p\,dx \right\}^{1/(p-1)} \tag{4.1.2}$$

The Fourier reciprocity holds in the sense that

$$F(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} f(t)\frac{e^{ixt}-1}{it}\,dt, \tag{4.1.3}$$

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} F(t)\frac{e^{-ixt}-1}{-it}\,dt, \tag{4.1.4}$$

almost everywhere.

As in the L^2 case, we might replace $F(x,a)$ by

$$F(x,a,b) = \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{b} f(t)e^{ixt}\,dt,$$

where $a \to \infty$, $b \to \infty$, in any manner.

4.2. The most obvious source of such results is in the formulae (2.1.11). If k is an integer, the transform of $\{F(x)\}^k$ is formally $(2\pi)^{-\frac{1}{2}k+\frac{1}{2}}\phi_k(x)$, where

$$\phi_k(x) = \int_{-\infty}^{\infty} f(u_{k-1})\,du_{k-1} \cdots \int_{-\infty}^{\infty} f(u_1)f(x-u_1-\ldots-u_{k-1})\,du_1.$$

We can deal with such integrals by means of the following lemmas.‡

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† Titchmarsh (2). ‡ See Hardy, Littlewood, and Pólya, *Inequalities*, pp. 198–203.

Lemma α. (Young's inequality.) *If $f(x)$ and $g(x)$ belong to $L^{1/(1-\lambda)}$ and $L^{1/(1-\mu)}$ respectively, where $\lambda > 0$, $\mu > 0$, $\lambda+\mu < 1$, then*

$$\left|\int_{-\infty}^{\infty} fg\,dx\right| \leqslant \left(\int_{-\infty}^{\infty} |f|^{1/(1-\lambda)}|g|^{1/(1-\mu)}\,dx\right)^{1-\lambda-\mu}\left(\int_{-\infty}^{\infty} |f|^{1/(1-\lambda)}\,dx\right)^{\mu}\left(\int_{-\infty}^{\infty} |g|^{1/(1-\mu)}\,dx\right)^{\lambda}.$$

Hölder's inequality for three functions is

$$\left|\int_{-\infty}^{\infty} \phi\psi\chi\,dx\right| \leqslant \left(\int_{-\infty}^{\infty} |\phi|^{1/\alpha}\,dx\right)^{\alpha}\left(\int_{-\infty}^{\infty} |\psi|^{1/\beta}\,dx\right)^{\beta}\left(\int_{-\infty}^{\infty} |\chi|^{1/\gamma}\,dx\right)^{\gamma},$$

where $\alpha+\beta+\gamma = 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Putting

$$|\phi|^{1/\alpha} = |\psi|^{1/\beta}|\chi|^{1/\gamma}, \qquad |\psi|^{1+\alpha/\beta} = |f|, \qquad |\chi|^{1+\alpha/\gamma} = |g|,$$

and $\gamma = \lambda$, $\beta = \mu$, the result follows.

Lemma β. *Let* $\quad \mathfrak{I}_p(f) = \left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}$

If $$\phi(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt,$$

then $$\mathfrak{I}_{1/(1-\lambda-\mu)}(\phi) \leqslant \mathfrak{I}_{1/(1-\lambda)}(f)\mathfrak{I}_{1/(1-\mu)}(g).$$

Young's inequality gives

$$|\phi(x)| \leqslant \left(\int_{-\infty}^{\infty} |f(t)|^{1/(1-\lambda)}|g(x-t)|^{1/(1-\mu)}\,dt\right)^{1-\lambda-\mu} \times$$
$$\cdot \times \{\mathfrak{I}_{1/(1-\lambda)}(f)\}^{\mu/(1-\lambda)}\{\mathfrak{I}_{1/(1-\mu)}(g)\}^{\lambda/(1-\mu)}.$$

Hence

$$\int_{-\infty}^{\infty} |\phi(x)|^{1/(1-\lambda-\mu)}\,dx \leqslant \int_{-\infty}^{\infty} |f(t)|^{1/(1-\lambda)}\,dt \int_{-\infty}^{\infty} |g(x-t)|^{1/(1-\mu)}\,dx \times$$
$$\times \{\mathfrak{I}_{1/(1-\lambda)}(f)\}^{\mu/(1-\lambda)(1-\lambda-\mu)}\{\mathfrak{I}_{1/(1-\mu)}(g)\}^{\lambda/(1-\mu)(1-\lambda-\mu)}$$
$$= \{\mathfrak{I}_{1/(1-\lambda)}(f)\}^{\frac{1}{1-\lambda}\left\{1+\frac{\mu}{1-\lambda-\mu}\right\}}\{\mathfrak{I}_{1/(1-\mu)}(g)\}^{\frac{1}{1-\mu}\left\{1+\frac{\lambda}{1-\lambda-\mu}\right\}},$$

and the result follows.

Lemma γ. *If $f(x)$ belongs to $L^{2k/(2k-1)}$, then $\phi_k(x)$ belongs to L^2, and*

$$\int_{-\infty}^{\infty} \{\phi_k(x)\}^2\,dx \leqslant \left(\int_{-\infty}^{\infty} |f(x)|^{2k/(2k-1)}\,dx\right)^{2k-1}$$

We have
$$\phi_k(x) = \int_{-\infty}^{\infty} f(t)\phi_{k-1}(x-t)\,dt,$$
and, applying the previous lemma repeatedly,
$$\mathfrak{J}_{1/(1-\frac{k}{2k})}(\phi_k) \leqslant \mathfrak{J}_{1/(1-\frac{k-1}{2k})}(\phi_{k-1})\mathfrak{J}_{1/(1-\frac{1}{2k})}(f)$$
$$\leqslant \mathfrak{J}_{1/(1-\frac{k-2}{2k})}(\phi_{k-2})\Big\{\mathfrak{J}_{1/(1-\frac{1}{2k})}(f)\Big\}^2 \leqslant \dots \leqslant \Big\{\mathfrak{J}_{1/(1-\frac{1}{2k})}(f)\Big\}^k,$$
the result stated.

4.3. Proof† of Theorem 74 for $p = 2k/(2k-1)$. Suppose first that $f(x)$ and $g(x)$ belong to L^2, and are zero outside a finite interval. Then
$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g(u_1)f(x-u_1)\,du_1$$
satisfies the same conditions, and (e.g. by Theorem 64) its transform is $F(x)G(x)$.

Repeating the argument $k-1$ times, and making all the functions equal, we see that the functions $\{F(x)\}^k$ and $(2\pi)^{-\frac{1}{2}k+\frac{1}{2}}\phi_k(x)$ are transforms of the class L^2. Hence
$$\int_{-\infty}^{\infty} |F(x)|^{2k}\,dx = \frac{1}{(2\pi)^{k-1}}\int_{-\infty}^{\infty} |\phi_k(x)|^2\,dx$$
$$\leqslant \frac{1}{(2\pi)^{k-1}}\left(\int_{-\infty}^{\infty} |f(x)|^{2k/(2k-1)}\,dx\right)^{2k-1}$$
This proves (4.1.2) for the special class of functions considered, and $p = 2k/(2k-1)$.

Now let $f(x)$ be any function of the class $L^{2k/(2k-1)}$. The function equal to $f(x)$ if $a < |x| < b$ and $|f(x)| \leqslant n$, and to 0 elsewhere, belongs to the special class. Applying the above result to it, and making $n \to \infty$, we obtain
$$\int_{-\infty}^{\infty} |F(x,b)-F(x,a)|^{2k}\,dx \leqslant \frac{1}{(2\pi)^{k-1}}\left\{\left(\int_{-b}^{-a} + \int_{a}^{b}\right)|f(x)|^{2k/(2k-1)}\,dx\right\}^{2k-1}.$$
The right-hand side tends to 0 as $a \to \infty$, $b \to \infty$. Hence $F(x,a)$ con-

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† The argument is analogous to that of W. H. Young for Fourier series. For a list of Young's papers see Zygmund's bibliography.

verges in mean, to $F(x)$ say. We have

$$\int_{-\infty}^{\infty} |F(x)|^{2k}\,dx = \lim_{a\to\infty} \int_{-\infty}^{\infty} |F(x,a)|^{2k}\,dx$$

$$\leqslant \lim_{a\to\infty} \frac{1}{(2\pi)^{k-1}} \left(\int_{-a}^{a} |f(x)|^{2k/(2k-1)}\,dx \right)^{2k-1}$$

$$= \frac{1}{(2\pi)^{k-1}} \left(\int_{-\infty}^{\infty} |f(x)|^{2k/(2k-1)}\,dx \right)^{2k-1}.$$

Also

$$\int_{0}^{\xi} F(x)\,dx = \lim_{a\to\infty} \int_{0}^{\xi} F(x,a)\,dx = \lim_{a\to\infty} \frac{1}{\sqrt{(2\pi)}} \int_{0}^{\xi} dx \int_{-a}^{a} f(t)e^{ixt}\,dt$$

$$= \lim_{a\to\infty} \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} f(t)\frac{e^{i\xi t}-1}{it}\,dt = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\frac{e^{i\xi t}-1}{it}\,dt,$$

so that

$$F(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} f(t)\frac{e^{ixt}-1}{it}\,dt$$

almost everywhere. Again, if $0 < \xi < a$,

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u,a)\frac{e^{-i\xi u}-1}{-iu}\,du = \frac{1}{2\pi} \int_{-a}^{a} f(t)\,dt \int_{-\infty}^{\infty} \frac{e^{-i\xi u}-1}{-iu} e^{iut}\,du$$

$$= \int_{0}^{\xi} f(t)\,dt,$$

the inversion being justified by the bounded convergence of the u-integral. Making $a \to \infty$, we obtain

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u)\frac{e^{-i\xi u}-1}{-iu}\,du = \int_{0}^{\xi} f(t)\,dt,$$

since $(e^{-i\xi u}-1)/(-iu)$ belongs to $L^{2k/(2k-1)}$. Hence

$$\frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{-\infty}^{\infty} F(u)\frac{e^{-ixu}-1}{-iu}\,du = f(x)$$

almost everywhere ($x > 0$). Similarly for $x < 0$.

4.4. Extension to general p. To extend the theorem to other values of p we first prove the corresponding theorem for trigonometrical polynomials.

LEMMA δ. *For any given numbers* c_m $(-n \leqslant m \leqslant n)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{m=-n}^{n} c_m e^{imx}\right|^{p'} dx \leqslant \Big(\sum_{m=-n}^{n} |c_m|^p\Big)^{1/(p-1)}. \tag{4.4.1}$$

We give two proofs, the original proof of Young and Hausdorff and a later one of Hardy and Littlewood.

(i)† Let $f(t)$ be a function of $L(-\pi,\pi)$, and let

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-imt}\,dt \quad (m = 0, \pm 1, \ldots). \tag{4.4.2}$$

We write
$$J_p(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p\,dt\right)^{1/p} \tag{4.4.3}$$

and
$$S_p(f) = \Big(\sum_{-\infty}^{\infty} |c_m|^p\Big)^{1/p} \tag{4.4.4}$$

We have to prove that, for any trigonometrical polynomial $f(t)$,

$$J_{p'} \leqslant S_p \quad (1 < p \leqslant 2). \tag{4.4.5}$$

If p is of the form $2k/(2k-1)$, this follows from the argument for sums parallel to that just given for integrals. If

$$f(x) = \sum c_m e^{imx}, \qquad g(x) = \sum \gamma_m e^{imx},$$

then
$$f(x)g(x) = \sum d_m e^{imx},$$

where
$$d_m = \sum c_r \gamma_{m-r}.$$

The analogue of Lemma β is therefore

$$S_{1/(1-\lambda-\mu)}(fg) \leqslant S_{1/(1-\lambda)}(f)S_{1/(1-\mu)}(g), \tag{4.4.6}$$

and, as in Lemma γ, it follows that

$$S_2(f^k) \leqslant \left\{S_{1/(1-\frac{1}{2k})}(f)\right\}^k. \tag{4.4.7}$$

But for any trigonometrical polynomial ϕ

$$S_2(\phi) = J_2(\phi). \tag{4.4.8}$$

Thus
$$S_2(f^k) = J_2(f^k) = \{J_{2k}(f)\}^k,$$

and (4.4.5) with $p = 2k/(2k-1)$ follows.

To extend it to other values of p we consider maximal polynomials, viz. those for which $J_{p'}^{p'}$ is a maximum for a given value of S_p^p, and a given n. Since S_p^p, $J_{p'}^{p'}$ are continuous functions with continuous partial derivatives with respect to the components x_m, y_m of the coefficients $c_m = x_m+iy_m$, maximal polynomials exist; and we can determine them by the ordinary method of the differential calculus. Let

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† Hausdorff (1).

$$\phi = \sum_{m=-n}^{n} (x_m^2+y_m^2)^{\frac{1}{2}p}, \quad \psi = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(t)|^{p'}\,dt.$$

Then the condition for a maximum is

$$\frac{\dfrac{\partial\psi}{\partial x_m}}{\dfrac{\partial\phi}{\partial x_m}} = \frac{\dfrac{\partial\psi}{\partial y_m}}{\dfrac{\partial\phi}{\partial y_m}} = \lambda \quad (m = -n,\ldots,n).$$

Hence
$$\frac{\dfrac{\partial\psi}{\partial x_m}+i\dfrac{\partial\psi}{\partial y_m}}{\dfrac{\partial\phi}{\partial x_m}+i\dfrac{\partial\phi}{\partial y_m}} = \lambda \quad (m = -n,\ldots,n).$$

Now†
$$\frac{\partial\phi}{\partial x_m}+i\frac{\partial\phi}{\partial y_m} = p(x_m+iy_m)(x_m^2+y_m^2)^{\frac{1}{2}p-1} = p|c_m|^{p-1}\operatorname{sgn} c_m.$$

Also
$$|f(t)|^2 = f(t)\bar{f}(t),$$
$$2|f(t)|\frac{\partial}{\partial x_m}|f(t)| = \bar{f}(t)\frac{\partial}{\partial x_m}f(t)+f(t)\frac{\partial}{\partial x_m}\bar{f}(t)$$
$$= \bar{f}(t)e^{imt}+f(t)e^{-imt},$$

and similarly $2|f(t)|\dfrac{\partial}{\partial y_m}|f(t)| = \bar{f}(t)ie^{imt}-f(t)ie^{-imt}$.

Hence
$$2|f(t)|\left(\frac{\partial}{\partial x_m}+i\frac{\partial}{\partial y_m}\right)|f(t)| = 2f(t)e^{-imt},$$
$$\left(\frac{\partial}{\partial x_m}+i\frac{\partial}{\partial y_m}\right)|f(t)| = e^{-imt}\operatorname{sgn} f(t).$$

Hence
$$\frac{\partial\psi}{\partial x_m}+i\frac{\partial\psi}{\partial y_m} = \frac{p'}{2\pi}\int\limits_{-\pi}^{\pi} |f(t)|^{p'-1}\operatorname{sgn} f(t)\,e^{-imt}\,dt.$$

Hence
$$\frac{p'}{2\pi}\int\limits_{-\pi}^{\pi} |f(t)|^{p'-1}\operatorname{sgn} f(t)\,e^{-imt}\,dt = \lambda p|c_m|^{p-1}\operatorname{sgn} c_m \quad (m = -n,\ldots,n). \tag{4.4.9}$$

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† $\operatorname{sgn} z = \dfrac{z}{|z|}$ $(z \neq 0)$, $\operatorname{sgn} 0 = 0$.

To find λ, multiply by $\overline{c_m}$ and sum. We obtain

$$\frac{p'}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p'}\,dt = \lambda p\sum|c_m|^p,$$

i.e.
$$p'J_{p'}^{p'} = \lambda pS_p^p. \tag{4.4.10}$$

Now (4.4.9) gives the first $2n+1$ Fourier coefficients of the function $|f(t)|^{p'-1}\operatorname{sgn}f(t)$. Hence Bessel's inequality for this function gives

$$\sum_{m=-n}^{n}\left|\frac{\lambda p}{p'}|c_m|^{p-1}\operatorname{sgn}c_m\right|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p'-1}\operatorname{sgn}f(t)|^2\,dt,$$

i.e.
$$\frac{\lambda^2p^2}{p'^2}\sum_{m=-n}^{n}|c_m|^{2p-2} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{2p'-2}\,dt,$$

i.e.
$$\lambda^2p^2S_{2p-2}^{2p-2} \leqslant p'^2J_{2p'-2}^{2p'-2}. \tag{4.4.11}$$

From (4.4.10) and (4.4.11) it follows that

$$\frac{J_{p'}^{p'}}{S_p^p} \leqslant \frac{J_{2p'-2}^{p'-1}}{S_{2p-2}^{p-1}} \tag{4.4.12}$$

for every maximal polynomial.

Let $$r' = 2p'-2, \qquad r = 2/(3-p).$$

Then it follows from Hölder's inequality that

$$S_r^{p'-1} \leqslant S_{2p-2}^{p-1}\,S_p^{p'-p}, \tag{4.4.13}$$

and (4.4.12) and (4.4.13) give

$$\left(\frac{J_{p'}}{S_p}\right)^{p'} \leqslant \left(\frac{J_{r'}}{S_r}\right)^{p'-1}. \tag{4.4.14}$$

Now suppose that (4.4.5) holds for $p' = r$ and all polynomials. Then it follows from (4.4.14) that it holds for $p' = \frac{1}{2}r'+1$ for maximal polynomials, and so *a fortiori* for all polynomials. We have already proved it for $p' = 2k$. Hence we deduce it in succession for

$$p' = k+1, \frac{k+3}{2}, \frac{k+7}{4}, \ldots,$$

i.e. for all rational numbers whose denominators are powers of 2. Since these numbers are everywhere dense, the general result now follows from the continuity of S_p and $J_{p'}$ as functions of p.

(ii)† We again consider maximal polynomials; but, instead of the general condition for a maximum of a function of many variables, we use the theorem that, in Hölder's inequality

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† Hardy and Littlewood (1). See also F. Riesz (1).

$$\sum a_m b_m \leqslant \left(\sum |a_m|^p\right)^{1/p} \left(\sum |b_m|^{p'}\right)^{1/p'},$$

the case of equality occurs only if the $|a_m|^p$ and $|b_m|^{p'}$ are proportional. Also this proof is independent of the lemmas of § 4.2.

We define S_p and J_p as before, and write

$$f_n(x) = \sum_{m=-n}^{n} c_m e^{imx}.$$

For given n and p, let the upper bound of $S_{p'}(f_n)/J_p(f)$ for all f be denoted by $M = M(n) = M_p(n)$; and let the upper bound of $J_{p'}(f_n)/S_p(f_n)$ for all sets of c_m be $M' = M'(n) = M_p'(n)$. We first show that these bounds exist for every n.

We may suppose on grounds of homogeneity that $S_{p'}(f_n) = 1$. Then $|c_m|^{p'} \geqslant 1/(2n+1)$ for some value of m. Hence

$$(2n+1)^{-1/p'} \leqslant |c_m| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx \leqslant J_p,$$

and so $$M_p(n) \leqslant (2n+1)^{1/p'}.$$

Again, let $$g(x) = |f_n(x)|^{p'-1} \operatorname{sgn} f_n(x),$$

and let $$\gamma_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx}\, dx.$$

Then

$$J_{p'}^{p'}(f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(x) g(x)\, dx = \sum_{m=-n}^{n} c_m \gamma_m \leqslant \sum_{m=-n}^{n} |c_m \gamma_m|$$
$$\leqslant S_p(f_n) S_{p'}(g_n) \leqslant M S_p(f_n) J_p(g) = M S_p(f_n) J_{p'}^{1/(p-1)}(f_n), \tag{4.4.15}$$

and, dividing by $J_{p'}^{1/(p-1)}(f_n)$, it follows that M' is finite, and $M' \leqslant M$.

Again, if $$h_n(x) = \sum_{m=-n}^{n} |c_m|^{p'-1} \operatorname{sgn} c_m e^{imx},$$

we have (by an obvious term-by-term integration)

$$S_{p'}^{p'}(f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) h_n(x)\, dx \leqslant J_p(f) J_{p'}(h_n) \leqslant M' J_p(f) S_p(h_n)$$
$$= M' J_p(f) S_{p'}^{p'-1}(f_n),$$

and hence $M \leqslant M'$. Hence in fact $M = M'$. The example $f(x) = 1$ shows that $M \geqslant 1$.

Suppose now that $f_n(x)$ is a polynomial for which the maximum M'

of $J_{p'}(f_n)/S_p(f_n)$ is attained (since it is a continuous function of the variables c_m, there is such a polynomial). Since $M = M'$, the extreme terms of the chain (4.4.15) are then equal, and so all the terms are equal. The case of Hölder's inequality used is therefore an equality, and hence

$$|c_m|^p = \lambda|\gamma_m|^{p'} \quad (-n \leqslant m \leqslant n),$$

where λ is independent of m. Hence

$$S_p^p(f_n) = \lambda S_{p'}^{p'}(g_n).$$

But, since f_n is a maximal polynomial,

$$S_p(f_n) = \frac{1}{M} J_{p'}(f_n) = \frac{1}{M} J_p^{p-1}(g) = M^{-p} S_{p'}^{p-1}(g_n),$$

the last step depending on the equality of the 5th and 6th terms of (4.4.15). Hence

$$\lambda = M^{-p^2} S_{p'}^{p^2-p-p'}(g_n).$$

Let

$$r' = 2p'-2, \qquad r = 2/(3-p).$$

Then by Bessel's inequality

$$S_2^2(g_n) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |g|^2\, dx = J_{r'}^{r'}(f_n) \leqslant M_r^{r'} S_r^{r'}(f_n)$$

$$= M_r^{r'} \lambda^{\frac{r'}{p}} \{S_{r/(p-1)}(g_n)\}^{\frac{r'}{p-1}} = M_r^{r'} M_p^{-pr'} \{S_{p'}(g_n)\}^{r'\left(p-1-\frac{1}{p-1}\right)} \{S_{r/(p-1)}(g_n)\}^{\frac{r'}{p-1}}.$$

Since

$$\sum |\gamma_m|^{\frac{r}{p-1}} \leqslant \left(\sum |\gamma_m|^2\right)^{\frac{p-1}{3-p}} \left(\sum |\gamma_m|^{\frac{p}{p-1}}\right)^{\frac{2(2-p)}{3-p}},$$

the product of these S-terms on the right-hand side does not exceed $S_2^2(g_n)$. Hence

$$1 \leqslant M_r^{r'} M_p^{-pr'},$$

and so

$$M_r \geqslant M_p^p \geqslant M_p.$$

We can now repeat the argument with p replaced by r, and r by $s = 2/(3-r)$; and so on indefinitely. We thus obtain a sequence of values of p tending to 1 (since $r'-2 = 2(p'-2)$, etc.) through which M_p is non-decreasing. But

$$M_p(n) \leqslant (2n+1)^{1/p'} \to 1$$

as $p \to 1$, $p' \to \infty$, for a fixed n. Hence $M_p(n) = 1$.

4.5. We can now prove Theorem 74 by the method used in § 3.2. Let $f(x)$ belong to L^p, $1 < p < 2$, and define a_ν and $\Phi_n(x)$ as in § 3.2. Then

$$\int_{-\pi\lambda}^{\pi\lambda} |\Phi_n(x)|^{p'}\, dx = \lambda \int_{-\pi}^{\pi} \left|\sum_{\nu=-n}^{n} a_\nu e^{i\nu x}\right|^{p'} dx \leqslant 2\pi\lambda \left(\sum_{\nu=-n}^{n} |a_\nu|^p\right)^{1/(p-1)}$$

by the above lemma; and

$$|a_\nu|^p \leqslant \int\limits_{\nu/\lambda}^{(\nu+1)/\lambda} |f(x)|^p\,dx \left(\int\limits_{\nu/\lambda}^{(\nu+1)/\lambda} dx\right)^{p-1} = \frac{1}{\lambda^{p-1}} \int\limits_{\nu/\lambda}^{(\nu+1)/\lambda} |f(x)|^p\,dx.$$

Hence $$\int\limits_{-\pi\lambda}^{\pi\lambda} |\Phi_n(x)|^{p'}\,dx \leqslant 2\pi\left(\int\limits_{-b}^{b} |f(x)|^p\,dx\right)^{1/(p-1)}$$

It follows as in § 3.2 that

$$\int\limits_{-\infty}^{\infty} |F(x,b)-F(x,a)|^{p'}\,dx \leqslant (2\pi)^{1-\frac{1}{2}p'}\left\{\left(\int\limits_{-b}^{-a} + \int\limits_{a}^{b}\right)|f(x)|^p\,dx\right\}^{1/(p-1)}$$

Hence $F(x,a)$ converges in mean as $a \to \infty$, to $F(x)$ say, with exponent p'. The remainder of the proof is the same as in the special case where $p' = 2k$.

Still another proof of Theorem 74 can be obtained from a general theorem of M. Riesz on functional operations. See Zygmund, § 9.2.

4.6. The Parseval formula.

THEOREM 75. *If $f(x)$ and $G(x)$ belong to $L^p(-\infty,\infty)$, $1 < p < 2$, and $F(x)$ and $g(x)$ are their transforms, then* (2.1.1) *holds.*

We know that if $\phi(x)$ is L^p, and $\psi(x,a)$ converges in mean to $\psi(x)$ with exponent p', then†

$$\lim_{a\to\infty} \int \{\psi(x)-\psi(x,a)\}\phi(x)\,dx = 0. \tag{4.6.1}$$

Now

$$\int\limits_{-b}^{b} F(x,a)G(x)\,dx = \frac{1}{\sqrt{(2\pi)}} \int\limits_{-b}^{b} G(x)\,dx \int\limits_{-a}^{a} f(t)e^{ixt}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int\limits_{-a}^{a} f(t)\,dt \int\limits_{-b}^{b} G(x)e^{ixt}\,dx = \int\limits_{-a}^{a} f(t)g(-t,b)\,dt.$$

Making $a \to \infty$, and applying (4.6.1) to the left-hand side, we obtain

$$\int\limits_{-b}^{b} F(x)G(x)\,dx = \int\limits_{-\infty}^{\infty} f(t)g(-t,b)\,dt.$$

Making $b \to \infty$, and applying (4.6.1) to the right-hand side, we obtain (2.1.1).

There are also obvious extensions of Theorems 58–62.

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† Titchmarsh, *Theory of Functions*, § 12.53.

4.7. Theorems on resultants.

THEOREM 76. *If $f(x)$, $F(x)$ are transforms of L^p, $L^{p'}$, and $g(x)$, $G(x)$ of $L^{p'}$, L^p, then* (2.1.9) *are transforms in the sense that* (2.1.8) *holds for all values of x.*

Proof similar to that of Theorem 64.

THEOREM 77. *If $f(x)$, $F(x)$ are transforms of L^p, $L^{p'}$, and $g(x)$ is L, then* (2.1.9) *are transforms of L^p, $L^{p'}$.*

Proof similar to that of Theorem 65.

THEOREM 78. *Let $f(x)$, $F(x)$ be transforms of L^p, $L^{p'}$, and $g(x)$, $G(x)$ of L^q, $L^{q'}$, where*

$$\frac{1}{p}+\frac{1}{q}>1. \tag{4.7.1}$$

Then

$$F(x)G(x), \qquad \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

are transforms of classes $L^{P'}$, L^P respectively, where

$$P=\frac{pq}{p+q-pq}.$$

That the resultant of f and g belongs to L^P follows from Lemma β of § 4.2, with $1-\lambda = 1/p$, $1-\mu = 1/q$. That FG belongs to $L^{P'}$ follows at once from Hölder's inequality in the form

$$\int |FG|^{P'}\,dx \leqslant \left(\int |F|^{p'}\,dx\right)^{P'/p'}\left(\int |G|^{q'}\,dx\right)^{P'/q'}.$$

The condition (4.7.1) implies that $p<q'$, $q<p'$. Suppose that $p\leqslant q$. Then $p<p'$, i.e. $p<2$.

Suppose that $1<P'\leqslant 2$. Then FG has a transform, the integral of which is

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(u)G(u)\frac{e^{-ixu}-1}{-iu}\,du.$$

Now $G(u)(e^{-ixu}-1)/(-iu)$ belongs to L and to $L^{q'}$, and so to L^p; and, by Theorem 74,

$$\int_{y}^{x+y} g(\xi)\,d\xi = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(u)\frac{e^{-ixu}-1}{-iu}e^{-iyu}\,du,$$

i.e. it is the transform of $G(u)(e^{-ixu}-1)/(-iu)$. Hence, by Theorem 76,

$$\int\limits_{-\infty}^{\infty} F(u)G(u)\frac{e^{-ixu}-1}{-iu}\,du = \int\limits_{-\infty}^{\infty} f(y)\,dy \int\limits_{-y}^{x-y} g(\xi)\,d\xi$$

$$= \int\limits_{-\infty}^{\infty} f(y)\,dy \int\limits_{0}^{x} g(u-y)\,du = \int\limits_{0}^{x} du \int\limits_{-\infty}^{\infty} f(y)g(u-y)\,dy,$$

and the result follows on differentiating.

Suppose next that $1 < P < 2$. Then

$$\frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

has a transform, the integral of which is

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu}\,du \int\limits_{-\infty}^{\infty} f(y)g(u-y)\,dy.$$

This is the limit as $a \to \infty$ of

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu}\,du \int\limits_{-a}^{a} f(y)g(u-y)\,dy$$

$$\left(\text{since } \int\limits_{-\infty}^{\infty} \ldots\,dy = \underset{(P)}{\text{l.i.m.}} \int\limits_{-a}^{a}\right)$$

$$= \frac{1}{2\pi} \int\limits_{-a}^{a} f(y)\,dy \int\limits_{-\infty}^{\infty} g(u-y)\frac{e^{ixu}-1}{iu}\,du,$$

and by the Parseval formula (for q, q') the inner integral is equal to

$$\sqrt{(2\pi)} \int\limits_{0}^{x} G(v)e^{ivy}\,dv.$$

Hence we obtain

$$\frac{1}{\sqrt{(2\pi)}} \int\limits_{-a}^{a} f(y)\,dy \int\limits_{0}^{x} G(v)e^{ivy}\,dv = \frac{1}{\sqrt{(2\pi)}} \int\limits_{0}^{x} G(v)\,dv \int\limits_{-a}^{a} f(y)e^{ivy}\,dy$$

$$\to \int\limits_{0}^{x} G(v)F(v)\,dv,$$

since $G(v)$ belongs to L^p over $(0, x)$. Hence the result.

If $p = 1$ and $q = 1$, then $P = 1$; see Theorem 41.

4.8. Another extension of Plancherel's theorem.

We shall next obtain a generalization of Plancherel's theorem in a different direction, due to Hardy and Littlewood.†

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† Hardy and Littlewood (1).

THEOREM 79. *If* $|f(x)|^q x^{q-2}$ $(q > 2)$ *belongs to* $L(-\infty,\infty)$, *then* $F(x)$, *the transform of* $f(x)$, *exists, and belongs to* L^q; *and*

$$\int_{-\infty}^{\infty} |F(x)|^q\,dx \leqslant K(q) \int_{-\infty}^{\infty} |f(x)|^q |x|^{q-2}\,dx.$$

(i) Consider the case $q = 4$.

Suppose first that $f(x)$ belongs to L^2, and vanishes outside a finite interval. Then $F(x)$ is L^2 and bounded, and $\sqrt{(2\pi)}\{F(x)\}^2$ is the transform of

$$\phi(x) = \int_{-\infty}^{\infty} f(y)f(x-y)\,dy,$$

which also belongs to L^2. Hence

$$2\pi \int_{-\infty}^{\infty} |F(x)|^4\,dx = \int_{-\infty}^{\infty} |\phi(x)|^2\,dx.$$

Now $f(x) = |x|^{-\frac{1}{2}}g(x)$, where $g(x)$ belongs to L^4. Hence

$$\phi(x) = \int_{-\infty}^{\infty} \frac{g(y)}{|y|^{\frac{1}{2}}}\frac{g(x-y)}{|x-y|^{\frac{1}{2}}}\,dy = \int_{-\infty}^{\infty} \frac{g(y)}{|y|^{\frac{1}{8}}}\frac{g(x-y)}{|x-y|^{\frac{1}{8}}}\frac{1}{|y|^{\frac{3}{8}}|x-y|^{\frac{3}{8}}}\,dy,$$

$$|\phi(x)|^2 \leqslant \int_{-\infty}^{\infty} \frac{|g(y)|^2}{|y|^{\frac{1}{4}}}\frac{|g(x-y)|^2}{|x-y|^{\frac{1}{4}}}\,dy \int_{-\infty}^{\infty} \frac{dy}{|y|^{\frac{3}{4}}|x-y|^{\frac{3}{4}}}$$

$$= \frac{A}{|x|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{|g(y)|^2}{|y|^{\frac{1}{4}}}\frac{|g(x-y)|^2}{|x-y|^{\frac{1}{4}}}\,dy.$$

Hence

$$\int_{-\infty}^{\infty} |\phi(x)|^2\,dx \leqslant A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|g(y)|^2|g(x-y)|^2}{|x|^{\frac{1}{2}}|y|^{\frac{1}{4}}|x-y|^{\frac{1}{4}}}\,dxdy$$

$$= A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|g(y)|^2|y|^{\frac{1}{8}}}{|x|^{\frac{1}{4}}|x-y|^{\frac{3}{8}}}\,\frac{|g(x-y)|^2|x-y|^{\frac{1}{8}}}{|x|^{\frac{1}{4}}|y|^{\frac{3}{8}}}\,dxdy$$

$$< A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\frac{|g(y)|^4|y|^{\frac{1}{4}}}{|x|^{\frac{1}{2}}|x-y|^{\frac{3}{4}}} + \frac{|g(x-y)|^4|x-y|^{\frac{1}{4}}}{|x|^{\frac{1}{2}}|y|^{\frac{3}{4}}}\right\}dxdy$$

$$= A \int_{-\infty}^{\infty} |g(y)|^4\,dy \int_{-\infty}^{\infty} \frac{|y|^{\frac{1}{4}}}{|x|^{\frac{1}{2}}|x-y|^{\frac{3}{4}}}\,dx = A \int_{-\infty}^{\infty} |g(y)|^4\,dy.$$

Hence $$\int_{-\infty}^{\infty} |F(x)|^4\,dx < A \int_{-\infty}^{\infty} |f(x)|^4 x^2\,dx.$$

The proof now follows the usual lines. Let $f(x)$ be any function such that $\{f(x)\}^4x^2$ is L. Approximating to $f(x)$ over (a,b) by a sequence of functions of the special type, we prove as in § 4.3 that

$$\int_{-\infty}^{\infty} |F(x,a)-F(x,b)|^4\,dx < A\left(\int_{-b}^{-a} + \int_{a}^{b}\right)|f(x)|^4x^2\,dx.$$

Hence $F(x,a)$ converges in mean with exponent 4, and the theorem follows in the usual way.

(ii) It is possible to prove the theorem when q is any even integer by an extension of the above method, but, as in the Young-Hausdorff theorem, the other values remain to be filled in.

The simplest procedure is to begin by proving the corresponding result for series, and we shall quote this from Zygmund.† The case we require is that if $f(x)$ has the Fourier coefficients c_m, then

$$\int_{-\pi}^{\pi} |f(x)|^q\,dx \leqslant K(q)\sum_{-\infty}^{\infty} |c_m|^q(|m|+1)^{q-2}.$$

Defining a_ν and $\Phi_n(x)$ as before, it follows that

$$\int_{-\pi\lambda}^{\pi\lambda} |\Phi_n(x)|^q\,dx = \lambda\int_{-\pi}^{\pi}\left|\sum_{\nu=-n}^{n} a_\nu e^{i\nu x}\right|^q dx$$
$$\leqslant \lambda K(q)\sum_{\nu=-n}^{n} |a_\nu|^q(|\nu|+1)^{q-2}.$$

If $\nu \geqslant 1$,
$$|a_\nu|^q \leqslant \frac{1}{\lambda^{q-1}}\int_{\nu/\lambda}^{(\nu+1)/\lambda} |f(x)|^q\,dx \leqslant \frac{1}{\lambda\nu^{q-2}}\int_{\nu/\lambda}^{(\nu+1)/\lambda} |f(x)|^q x^{q-2}\,dx.$$

There is a similar inequality for $\nu \leqslant -2$; and

$$|a_0| = \left|\int_0^{1/\lambda} f(x)x^{1-2/q}x^{2/q-1}\,dx\right|$$
$$\leqslant \left(\int_0^{1/\lambda} |f(x)|^q x^{q-2}\,dx\right)^{1/q}\left(\int_0^{1/\lambda} x^{(2/q-1)q/(q-1)}\,dx\right)^{1-1/q} = o(\lambda^{-1/q}),$$

and similarly for a_{-1}. Hence, making $\lambda \to \infty$, we obtain

$$\int_{-\infty}^{\infty} |F(x,b)-F(x,a)|^q\,dx < K(q)\left(\int_{-b}^{-a} + \int_{a}^{b}\right)|f(x)|^q|x|^{q-2}\,dx.$$

The theorem now follows as in previous cases.

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† Zygmund, *Trigonometrical Series*, § 9.4.

4.9. THEOREM 80.[†] *If $f(x)$ belongs to L^p $(1 < p < 2)$, then*

$$\int_{-\infty}^{\infty} |F(x)|^p |x|^{p-2}\,dx \leqslant K(p) \int_{-\infty}^{\infty} |f(x)|^p\,dx.$$

Let $g(x)$ be a function of $L^{p'}$, vanishing outside a finite interval (a,b), where $a > 0$. Then it also belongs to L^p. Hence, by Theorem 75,

$$\int_{-\infty}^{\infty} F(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)G(x)\,dx.$$

Also, by Theorem 79, with $q = p'$,

$$\int_{-\infty}^{\infty} |G(x)|^{p'}\,dx \leqslant K(p) \int_{-\infty}^{\infty} |g(x)|^{p'}|x|^{p'-2}\,dx.$$

Hence

$$\left|\int_{-\infty}^{\infty} F(x)g(x)\,dx\right| \leqslant \left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |G(x)|^{p'}\,dx\right)^{1/p'}$$

$$\leqslant K(p)\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |g(x)|^{p'}|x|^{p'-2}\,dx\right)^{1/p'}$$

Let $\qquad g(x) = |F(x)|^{p-1}\operatorname{sgn}\bar{F}(x)|x|^{p-2} \quad (a \leqslant x \leqslant b).$

Then

$$\int_{a}^{b} |F(x)|^p x^{p-2}\,dx \leqslant K(p)\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}\left(\int_{a}^{b} |F(x)|^p x^{p-2}\,dx\right)^{1/p'},$$

and hence $\qquad \displaystyle\int_{a}^{b} |F(x)|^p x^{p-2}\,dx \leqslant K(p) \int_{-\infty}^{\infty} |f(x)|^p\,dx.$

Making $a \to 0$, $b \to \infty$, we obtain the desired result for the integral over $(0,\infty)$; similarly for the integral over $(-\infty,0)$.

4.10. Another case of the Parseval formula.

THEOREM 81. *Let $f(x)$ be L^p, and let $|G(x)|^{p'}|x|^{p'-2}$ be L, where $1 < p < 2$. Then if F and g are the transforms of f and G,* (2.1.1) *holds.*

The proof is similar to that of Theorem 75, but now

$$\int_{-b}^{b} \{F(x,a)-F(x)\}G(x)\,dx = \int_{-b}^{b} \{F(x,a)-F(x)\}|x|^{(p-2)/p}\,.\,G(x)|x|^{1-2/p'}\,dx$$

tends to 0 because, by Theorem 80, $F(x,a)|x|^{(p-2)/p}$ converges in mean to $F(x)|x|^{(p-2)/p}$, with exponent p. The proof concludes as before, but is justified by Theorem 79 instead of Theorem 74.

[†] Hardy and Littlewood (1).

4.11. Failure of Theorems 75 and 79 for $p > 2$. That the Young-Hausdorff theorem fails for $p > 2$ follows incidentally from Theorem 80. For, if $f(x)$ belongs to L^p, not only is $|F(x)|^{p'}$ integrable, but so is $|F(x)|^p|x|^{p-2}$; and hence so is

$$|F(x)|^r|x|^{r/p'-1} \quad (p \leqslant r \leqslant p'). \tag{4.11.1}$$

Call the class of functions with this property $L_1^{p'}$, so that $L_1^{p'}$ is a sub-set of $L^{p'}$

If $f(x)$ belongs to L^q $(q > 2)$, it does not necessarily belong to L_1^q, and is therefore not necessarily the transform of a function of $L^{q'}$.

However, we can show by means of examples that even if $f(x)$ belongs to L_1^q $(q > 2)$, $f(x)$ is not necessarily the transform of a function of the class $L^{q'}$. Presumably no condition which merely states the existence of an integral involving $|f(x)|$ is a sufficient condition for $f(x)$ to be the transform of a function of $L^{q'}$.

Consider the function† $(0 < a < 1,\ a < b)$

$$\begin{aligned} f(x) &= \sqrt{\left(\frac{2}{\pi}\right)} \int_{\to 0}^{1} t^{-a-1} \cos t^{-b} \cos xt\, dt \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{\to 0}^{1} \cos\left(xt + \frac{1}{t^b}\right) \frac{dt}{t^{a+1}} + \frac{1}{\sqrt{(2\pi)}} \int_{\to 0}^{1} \cos\left(xt - \frac{1}{t^b}\right) \frac{dt}{t^{a+1}} \\ &= \frac{1}{\sqrt{(2\pi)}} \{\phi(x) + \psi(x)\}. \end{aligned} \tag{4.11.2}$$

Let

$$\phi(x) = \int_{\to 0}^{\left(\frac{b}{x}-\xi\right)^{1/(b+1)}} + \int_{\left(\frac{b}{x}-\xi\right)^{1/(b+1)}}^{\left(\frac{b}{x}+\xi\right)^{1/(b+1)}} + \int_{\left(\frac{b}{x}+\xi\right)^{1/(b+1)}}^{1} = \phi_1 + \phi_2 + \phi_3,$$

where $\xi = o(1/x)$ as $x \to \infty$. Then

$$\phi_1 = \int \frac{d\sin(xt+t^{-b})}{xt^{a+1} - bt^{a-b}},$$

and here $(bt^{a-b} - xt^{a+1})^{-1}$ is positive, steadily increasing, and less than

$$\frac{t^{b-a}}{x\left(\frac{b}{x} - t^{b+1}\right)} \leqslant \frac{(b/x)^{\frac{b-a}{b+1}}}{x\xi}.$$

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† Titchmarsh (2).

Hence, by the second mean-value theorem,

$$\phi_1 = O\left(\xi^{-1}x^{\frac{a-2b-1}{b+1}}\right).$$

In ϕ_3, $(xt^{a+1}-bt^{a-b})^{-1}$ is positive and steadily decreasing, and we obtain the same result as for ϕ_1. Finally

$$|\phi_2| \leqslant \frac{\left(\frac{b}{x}+\xi\right)^{1/(b+1)}-\left(\frac{b}{x}-\xi\right)^{1/(b+1)}}{\left(\frac{b}{x}-\xi\right)^{\frac{a+1}{b+1}}} = O\left(\frac{x^{-1/(b+1)}x\xi}{x^{-\frac{a+1}{b+1}}}\right) = O\left(\xi x^{\frac{a+b+1}{b+1}}\right).$$

Taking $\xi = x^{-\frac{3b+2}{2b+2}}$, it follows that, as $x \to \infty$,

$$\phi(x) = O\left(x^{\frac{2a-b}{2b+2}}\right).$$

Again,
$$\psi(x) = \int \frac{d\sin(xt-t^{-b})}{xt^{a+1}+bt^{a-b}}.$$

Now $(xt^{a+1}+bt^{a-b})^{-1}$ increases steadily from 0 to a maximum of the form $Kx^{\frac{a-b}{b+1}}$, where K depends on a and b only, and then decreases steadily. Hence the second mean-value theorem gives

$$\psi(x) = O\left(x^{\frac{a-b}{b+1}}\right).$$

Hence as $x \to \infty$
$$f(x) = O\left(x^{\frac{2a-b}{2b+2}}\right),$$

and plainly $f(x) = O(1)$ as $x \to 0$. Hence, if q is a given number greater than 2, $f(x)$ belongs to L^q if b is large enough.

If $f(x)$ were the transform of a function $F(x)$ of $L^{q'}$, we should have

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}f(u)\,du.$$

If we can insert (4.11.2) for $f(u)$ and invert the order of integration, we obtain

$$F(x) = x^{-a-1}\cos x^{-b} \quad (0 < x < 1), \qquad 0 \quad (x > 1),$$

which does not belong to L^r for any $r \geqslant 1$. This gives the desired result.

The inversion is justified if we may invert

$$\int_0^{\to\infty} \frac{\sin xu}{u}\,du \int_0^1 t^{-a-1}\cos t^{-b}(1-\cos ut)\,dt,$$

and we may if

$$\lim_{\lambda\to\infty}\int\limits_0^1 t^{-a-1}\cos t^{-b}\,dt\int\limits_\lambda^{\to\infty}\frac{\sin xu(1-\cos ut)}{u}\,du=0.$$

Now

$$\left|\int\limits_\lambda^{\to\infty}\frac{\sin xu(1-\cos ut)}{u}\,du\right|=\tfrac{1}{2}\left|\int\limits_{\lambda x}^{\lambda(x+t)}\frac{\sin v}{v}\,dv-\int\limits_{\lambda|x-t|}^{\lambda x}\frac{\sin v}{v}\,dv\right|$$

$$\leqslant\tfrac{1}{2}\log\left|\frac{x+t}{x-t}\right|,$$

and the result follows from dominated convergence.

4.12. Special conditions. In this section we give two sufficient conditions of special kinds for $f(x)$ to be the transform of a function of L^p $(1<p<2)$.

THEOREM 82.† *Let $f(x)$ be even, positive non-increasing for $x>0$, $f(\infty)=0$, and let $\{f(x)\}^p x^{p-2}$ $(1<p<2)$ belong to $L(0,\infty)$. Then $F(x)$ belongs to L^p.*

Since $f(x)$ in non-increasing, and $f(\infty)=0$, the integral

$$F(x)=F_c(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^{\to\infty}f(y)\cos xy\,dy$$

converges for every $x>0$. Let

$$F(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^{1/x}f(y)\cos xy\,dy+\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{1/x}^{\to\infty}f(y)\cos xy\,dy$$

$$=F_1(x)+F_2(x).$$

By the second mean-value theorem

$$F_2(x)=\sqrt{\left(\frac{2}{\pi}\right)}f\left(\frac{1}{x}\right)\int\limits_{1/x}^{\xi}\cos xy\,dy=\sqrt{\left(\frac{2}{\pi}\right)}f\left(\frac{1}{x}\right)\frac{\sin x\xi-\sin 1}{x}.$$

Hence
$$|F_2(x)|\leqslant 2\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{x}f\left(\frac{1}{x}\right),$$

and
$$\int\limits_0^\infty|F_2(x)|^p\,dx<A\int\limits_0^\infty\left\{\frac{1}{x}f\left(\frac{1}{x}\right)\right\}^p dx=A\int\limits_0^\infty\{f(t)\}^p t^{p-2}\,dt,$$

---

† See Hardy and Littlewood (3).

which is finite by hypothesis. Also

$$|F_1(x)| \leqslant \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{1/x} f(y)\,dy,$$

and we have to prove that this is L^p; or, what is the same thing, that

$$x^{-2/p} \int_0^x f(y)\,dy$$

is L^p. We are given that $f(x) = g(x)x^{2/p-1}$, where $g(x)$ is L^p. Hence

$$x^{-2/p} \int_0^x f(y)\,dy = x^{-2/p} \int_0^x g(y)y^{2/p-1}\,dy \leqslant x^{-1} \int_0^x g(y)\,dy,$$

which is L^p, by a theorem of Hardy.† This proves the theorem.

Incidentally it must follow from our hypothesis that $f(x)$ belongs to $L^{p'}$; and in fact

$$K > \int_{\frac{1}{2}x}^{x} f^p(t)t^{p-2}\,dt \geqslant f^p(x)(\tfrac{1}{2}x)^{p-2}\tfrac{1}{2}x,$$

$$f(x) < Kx^{-(p-1)/p},$$

$$|f(x)|^{p'} < K|f(x)|^p x^{p-2}.$$

Theorem 83. *Let $f(x)$ be the integral of order $(2-p)/p$ of a function $\phi(x)$ of L^p. Then $F_c(x)$ exists and belongs to L^p.*

Let
$$\Phi_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \phi(t)\sin(xt+\pi/p)\,dt.$$

Then, by Theorem 80, $x^{1-2/p}\Phi_a(x)$ converges in mean (p) to $g(x)$ say. Let $G_c(x)$ be the cosine transform of $g(x)$. Then $G_c(x)$ belongs to $L^{p'}$. Also

$$\int_0^y G_c(x)\,dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin xy}{x} g(x)\,dx$$

$$= \lim_{a\to\infty} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin xy}{x} x^{1-2/p}\Phi_a(x)\,dx$$

$$= \lim_{a\to\infty} \frac{2}{\pi} \int_0^\infty \frac{\sin xy}{x^{2/p}}\,dx \int_0^a \phi(t)\sin(xt+\pi/p)\,dt$$

$$= \lim_{a\to\infty} \frac{2}{\pi} \int_0^a \phi(t)\,dt \int_0^\infty \frac{\sin xy \sin(xt+\pi/p)}{x^{2/p}}\,dx,$$

the inversion being justified by uniform convergence.

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† See Titchmarsh, *Theory of Functions*, p. 396.

The inner integral is

$$\frac{\pi}{2\Gamma(2/p)}(y-t)^{2/p-1} \quad (t<y), \qquad 0 \quad (t>y).$$

Hence

$$\int_0^y G_c(x)\,dx = \frac{1}{\Gamma(2/p)}\int_0^y (y-t)^{2/p-1}\phi(t)\,dt = \int_0^y f(x)\,dx,$$

by hypothesis. Hence $G_c(x) = f(x)$ almost everywhere, and $g(x) = F_c(x)$ belongs to L^p.

4.13. Lipschitz conditions. In this section we shall give a condition of quite a different kind for a function to have a transform belonging to certain L-classes. The analysis originated with theorems of Bernstein and Szasz on Fourier series.†

THEOREM 84. *Let $f(x)$ belong to L^p $(1<p\leqslant 2)$, and let*

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^p\,dx = O(h^{\alpha p}) \quad (0<\alpha\leqslant 1) \quad (4.13.1)$$

as $h\to 0$. Then $F(x)$ belongs to L^β for

$$\frac{p}{p+\alpha p-1} < \beta \leqslant \frac{p}{p-1}.$$

For a fixed h the transform of $f(x+h)$ is $e^{-ixh}F(x)$. Hence the transform of $f(x+h)-f(x-h)$, as a function of x, is $-2i\sin xhF(x)$. Hence

$$\int_{-\infty}^{\infty} |2\sin xhF(x)|^{p'}\,dx < K(p)\left\{\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^p\,dx\right\}^{1/(p-1)}$$
$$< K(p)h^{\alpha p'}.$$

Since $|\sin xh| > Axh$ for $x\leqslant 1/h$, the left-hand side is greater than

$$A\int_0^{1/h} x^{p'}h^{p'}|F(x)|^{p'}\,dx.$$

Hence $$\int_0^{1/h} x^{p'}|F(x)|^{p'}\,dx = O(h^{(\alpha-1)p'}).$$

Let $$\phi(\xi) = \int_1^{\xi} |xF(x)|^\beta\,dx.$$

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† See Titchmarsh (12).

Then, if $\beta < p'$,

$$\phi(\xi) \leqslant \left(\int\limits_1^\xi |xF(x)|^{p'}\,dx\right)^{\beta/p'}\left(\int\limits_1^\xi dx\right)^{1-\beta/p'} = O(\xi^{1-\alpha\beta+\beta/p}).$$

Hence

$$\int\limits_1^\xi |F(x)|^\beta\,dx = \int\limits_1^\xi x^{-\beta}\phi'(x)\,dx = \xi^{-\beta}\phi(\xi)+\beta\int\limits_1^\xi x^{-\beta-1}\phi(x)\,dx$$

$$= O(\xi^{1-\beta-\alpha\beta+\beta/p})+O\left(\int\limits_1^\xi x^{-\beta-\alpha\beta+\beta/p}\,dx\right) = O(\xi^{1-\beta-\alpha\beta+\beta/p}),$$

and this is bounded as $\xi \to \infty$ if $1-\beta-\alpha\beta+\beta/p \leqslant 0$, i.e. if

$$\beta > \frac{p}{p+\alpha p-1}.$$

Similarly for the integral over $(-\xi, -1)$. This proves the theorem.

A particular case, corresponding to the original theorem of Bernstein, is that if the condition is satisfied with $\alpha > 1/p$, then $F(x)$ belongs to $L(0,\infty)$, so that the Fourier integral

$$\int\limits_{-\infty}^{\infty} F(t)e^{ixt}\,dt$$

is absolutely convergent for all values of x.

To show that the range for β in the above theorem cannot be extended, consider the even function

$$f(x) = \frac{1}{x^a+x} \quad (x > 0),$$

where $0 < a < 1/p$. For $x > 2h$

$$|f(x+h)-f(x-h)| = 2h|f'(x+\theta h)| \quad (-1 < \theta < 1)$$
$$\leqslant 2h|f'(x-h)| \leqslant 2h|f'(\tfrac{1}{2}x)|,$$

since $|f'(x)|$ is positive and steadily decreasing. Hence

$$\int\limits_{2h}^{\infty} |f(x+h)-f(x-h)|^p\,dx = O\left\{h^p\int\limits_{2h}^{\infty} |f'(\tfrac{1}{2}x)|^p\,dx\right\}$$

$$= O\left\{h^p\left(\int\limits_{2h}^{1} x^{-(a+1)p}\,dx + \int\limits_1^\infty x^{-2}\,dx\right)\right\} = O(h^{1-ap}).$$

Also

$$\int\limits_0^{2h} |f(x+h)-f(x-h)|^p\,dx = O\left(\int\limits_0^{2h} \frac{dx}{|x-h|^{ap}}\right) = O(h^{1-ap}).$$

The conditions of the theorem are therefore satisfied with $\alpha = 1/p-a$. Hence $F(x)$ belongs to L^{β} for $\beta > 1/(1-a)$. But† $F(x) \sim Kx^{a-1}$ as $x \to \infty$, so that $F(x)$ does not belong to $L^{1/(1-a)}$.

In the case $\alpha < 1$, $p = 2$ it is possible to put the theorem into a form in which it is reversible.

THEOREM 85. *If $f(x)$ belongs to L^2, the conditions*

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^2\,dx = O(|h|^{2\alpha}) \quad (0 < \alpha < 1), \tag{4.13.2}$$

$$\left(\int_{-\infty}^{-X} + \int_{X}^{\infty}\right)\{F(x)\}^2\,dx = O(X^{-2\alpha}) \quad (X \to \infty) \tag{4.13.3}$$

are equivalent.

Instead of an inequality we now obtain

$$\int_{-\infty}^{\infty} 4\sin^2 xh\,|F(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^2\,dx. \tag{4.13.4}$$

Suppose that (4.13.2) holds. Then (4.13.4) gives

$$\int_{1/(2h)}^{1/h} |F(x)|^2\,dx < A\int_{0}^{\infty} \sin^2 xh\,|F(x)|^2\,dx = O(h^{2\alpha}).$$

Hence

$$\int_{X}^{\infty} \{F(x)\}^2\,dx = \int_{X}^{2X} + \int_{2X}^{4X} + \dots = O\{X^{-2\alpha}+(2X)^{-2\alpha}+\dots\} = O(X^{-2\alpha}),$$

and similarly for $(-\infty, -X)$.

On the other hand, if (4.13.3) holds, then writing

$$\phi(X) = \int_{X}^{\infty} \{F(x)\}^2\,dx,$$

$$\int_{0}^{X} x^2\{F(x)\}^2\,dx = \int_{0}^{X} -x^2\phi'(x)\,dx = -X^2\phi(X)+2\int_{0}^{X} x\phi(x)\,dx$$
$$\leqslant 2\int_{0}^{X} O(x^{1-2\alpha})\,dx = O(X^{2-2\alpha}).$$

Hence

$$\int_{-\infty}^{\infty} \sin^2 xh\,\{F(x)\}^2\,dx = O\left(h^2\int_{-1/h}^{1/h} \{F(x)\}^2\,dx\right)+O\left(\int_{-\infty}^{-1/h} + \int_{1/h}^{\infty} \{F(x)\}^2\,dx\right)$$
$$= O(h^{2\alpha}),$$

and (4.13.2) now follows from (4.13.4).

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† See e.g. Theorems 126–7 below.

Since, if $\beta < 2$,

$$\int\limits_X^{2X} |F(x)|^\beta\,dx \leqslant \left(\int\limits_X^{2X} \{F(x)\}^2\,dx\right)^{\frac{1}{2}\beta} \left(\int\limits_X^{2X} dx\right)^{1-\frac{1}{2}\beta}$$
$$= O(X^{-\alpha\beta})O(X^{1-\frac{1}{2}\beta}) = O(X^{1-\alpha\beta-\frac{1}{2}\beta}),$$

it follows again that $F(x)$ belongs to L^β if $\beta > 1/(\alpha+\frac{1}{2})$, the case $p = 2$ of the above theorem. But this last step is, of course, not reversible.

4.14. Mellin transforms of the class L^p. Let us denote by $\mathfrak{L}^p$ the class of functions $f(x)$ such that

$$\int\limits_0^\infty |f(x)|^p \frac{dx}{x} < \infty.$$

Then we have

THEOREM 86. *If $\mathfrak{F}(k+it)$ belongs to L^p $(1 < p < 2)$, then its Mellin transform $f(x)$ exists, and $x^k f(x)$ belongs to $\mathfrak{L}^{p'}$.*

If $x^k f(x)$ belongs to $\mathfrak{L}^p$, then the Mellin transform $\mathfrak{F}(s)$ of $f(x)$ exists, and $\mathfrak{F}(k+it)$ belongs to $L^{p'}$.

THEOREM 87. *If $\mathfrak{F}(k+it)$ belongs to L^p, and $x^{1-k}g(x)$ to $\mathfrak{L}^p$, then*

$$\frac{1}{2\pi i} \int\limits_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\mathfrak{G}(1-s)\,ds = \int\limits_0^\infty f(x)g(x)\,dx.$$

These are readily obtained by transformation from Theorems 74 and 75.

THEOREM 88. *If $\mathfrak{F}(k+iv)$, $x^k f(x)$ are Mellin transforms of L^p, $\mathfrak{L}^{p'}$, and $x^{s-k}g(x)$, $\mathfrak{G}(s-k-iv)$ of $\mathfrak{L}^{p'}$, L^p, then* (2.1.15) *holds.*

THEOREM 89. *If $\mathfrak{F}(k+iv)$, $x^k f(x)$ are Mellin transforms of L^p, $\mathfrak{L}^{p'}$, and $\mathfrak{G}(s-k-iv)$, $x^{s-k}g(x)$ of L^q, $\mathfrak{L}^{q'}$, then* (2.1.16) *are Mellin transforms of $\mathfrak{L}^{P'}$, L^P.*

Note. It has recently been proved by Zygmund (2) that *if $f(x)$ is L^p, $1 < p < 2$, then* (3.11.2) *holds almost everywhere,* no logarithmic factor being required. If $f(x)$ satisfies the condition of Theorem 79, then $f(x)\log x$ is $L^2(1,\infty)$ (apply Hölder's inequality to the integral over $(2^n, 2^{n+1})$). Hence (3.11.2) holds almost everywhere by Theorem 62.

V

CONJUGATE INTEGRALS; HILBERT TRANSFORMS

5.1. Conjugate integrals. FOURIER'S integral formula may be written in the form

$$f(x) = \int_0^\infty \{a(t)\cos xt + b(t)\sin xt\}\, dt, \tag{5.1.1}$$

where

$$a(t) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos ut\, du, \qquad b(t) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin ut\, du. \tag{5.1.2}$$

The integral in (5.1.1) is, formally, the limit as $y \to 0$ of

$$\int_0^\infty \{a(t)\cos xt + b(t)\sin xt\}e^{-yt}\, dt = U(x,y) \tag{5.1.3}$$

say; and this is the real part of

$$\int_0^\infty \{a(t) - ib(t)\}e^{itz}\, dt = \Phi(z) \tag{5.1.4}$$

say, where $z = x+iy$.

The imaginary part of $\Phi(z)$ is

$$-\int_0^\infty \{b(t)\cos xt - a(t)\sin xt\}e^{-yt}\, dt = V(x,y) \tag{5.1.5}$$

say. Writing $-V(x,0) = g(x)$, we obtain

$$g(x) = \int_0^\infty \{b(t)\cos xt - a(t)\sin xt\}\, dt \tag{5.1.6}$$

$$= \frac{1}{\pi}\int_0^\infty dt \int_{-\infty}^\infty \sin(u-x)t f(u)\, du. \tag{5.1.7}$$

The integral (5.1.7) is called the *allied integral* of Fourier's integral. It is obtained formally from (5.1.1) by replacing a by b and b by $-a$.

Repeating the process, we return to minus the original integral. The relation between $f(x)$ and $g(x)$ is thus skew-reciprocal, i.e. reciprocal apart from a minus sign.

Again, we have formally

$$a(t) = \frac{1}{\sqrt{(2\pi)}}\{F(t)+F(-t)\}, \qquad b(t) = \frac{1}{i\sqrt{(2\pi)}}\{F(t)-F(-t)\}.$$

Hence

$$g(x) = \frac{1}{i\sqrt{(2\pi)}}\int_0^\infty \{F(t)-F(-t)\}\cos xt\,dt - \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \{F(t)+F(-t)\}\sin xt\,dt$$

$$= \frac{1}{i\sqrt{(2\pi)}}\left\{\int_0^\infty F(t)e^{-ixt}\,dt - \int_0^\infty F(-t)e^{ixt}\,dt\right\}$$

$$= \frac{1}{i\sqrt{(2\pi)}}\int_{-\infty}^\infty F(t)\operatorname{sgn} t\, e^{-ixt}\,dt.$$

Thus
$$G(t) = -iF(t)\operatorname{sgn} t. \tag{5.1.8}$$

If $f(x)$ is even, $b(t) = 0$, and $g(x)$ is minus the sine transform of the cosine transform of $f(x)$; similarly, if $f(x)$ is odd, $g(x)$ is the cosine transform of the sine transform of $f(x)$.

Again, we have formally

$$g(x) = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\lambda dt\int_{-\infty}^\infty \sin(u-x)t f(u)\,du$$

$$= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{\to\infty}^\infty \frac{1-\cos\lambda(u-x)}{u-x}f(u)\,du$$

$$= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda t}{t}\{f(x+t)-f(x-t)\}\,dt.$$

If $f(x)$ is a sufficiently regular function, the part involving $\cos\lambda t$ will tend to 0 as $\lambda \to \infty$, and we shall have

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt; \tag{5.1.9}$$

and similarly
$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{g(x+t)-g(x-t)}{t}\,dt. \tag{5.1.10}$$

The reciprocity expressed by (5.1.9), (5.1.10) was first noticed by Hilbert, and the two functions so connected are called *Hilbert transforms*.

Equivalent formulae are

$$g(x) = \frac{1}{\pi}P\int_{-\infty}^\infty \frac{f(t)}{t-x}dt, \qquad f(x) = -\frac{1}{\pi}P\int_{-\infty}^\infty \frac{g(t)}{t-x}\,dt, \tag{5.1.11}$$

where P denotes a principal value at $t = x$.

Simple pairs of conjugate functions $f(x)$, $g(x)$, are

$$1\ (0 < x < a),\quad 0 \text{ elsewhere}, \qquad \frac{1}{\pi}\log\left|\frac{a+x}{a-x}\right|,$$

$$\frac{1}{1+x^2}, \qquad -\frac{x}{1+x^2},$$

$$\cos x, \qquad -\sin x,$$

and any number of such examples can be written down by starting with a suitable analytic function $\Phi(z)$. Examples from Chapter VII are

$$|x|^{-\nu}J_\nu(|x|), \qquad -\operatorname{sgn} x\,|x|^{-\nu}\mathbf{H}_\nu(|x|)$$

from (7.1.11) and (7.2.8);

$$\operatorname{sgn} x\,|x|^{\nu}J_\nu(|x|), \qquad -|x|^{\nu}Y_\nu(|x|)$$

from (7.11.2) and (7.11.3); and

$$J_0(2\sqrt{|x|}), \qquad -\operatorname{sgn} x\{(2/\pi)K_0(2\sqrt{|x|})+Y_0(2\sqrt{|x|})\}$$

from (7.11.2), with $\nu = 0$ and $x = \frac{1}{2}(u/a+a/u)$, and (7.12.8).

5.2. Conditions which would justify the above formalities directly would be extremely complicated. Actually the simplest rigorous argument gives the reciprocity in a slightly different form.

THEOREM 90.[†] *Let $f(x)$ belong to $L^2(-\infty,\infty)$. Then the formula*

$$g(x) = -\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty} f(t)\log\left|1-\frac{x}{t}\right|\,dt \tag{5.2.1}$$

defines almost everywhere a function $g(x)$, also belonging to $L^2(-\infty,\infty)$. The reciprocal formula

$$f(x) = \frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty} g(t)\log\left|1-\frac{x}{t}\right|\,dt \tag{5.2.2}$$

also holds almost everywhere; and

$$\int_{-\infty}^{\infty} \{f(x)\}^2\,dx = \int_{-\infty}^{\infty} \{g(x)\}^2\,dx. \tag{5.2.3}$$

If we could perform the differentiations under the integral signs, we should obtain the reciprocity in the form already given. We shall see later that this is possible; but we begin with the form to which the theory of Fourier transforms leads directly.

Let $F(x)$ be the Fourier transform of $f(x)$, $G(x) = -iF(x)\operatorname{sgn} x$, and $g(x)$ the transform of $G(x)$. Then

$$\int_{-\infty}^{\infty} |g(x)|^2\,dx = \int_{-\infty}^{\infty} |G(x)|^2\,dx = \int_{-\infty}^{\infty} |F(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx.$$

[†] Titchmarsh (5).

Also

$$g(x)=\frac{1}{\sqrt{(2\pi)}}\frac{d}{dx}\int\limits_{-\infty}^{\infty}G(y)\frac{e^{-ixy}-1}{-iy}\,dy=\frac{1}{\sqrt{(2\pi)}}\frac{d}{dx}\int\limits_{-\infty}^{\infty}F(y)\frac{e^{-ixy}-1}{|y|}\,dy.$$

The transform of $H(y)=(e^{-ixy}-1)/|y|$ is

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{e^{-ixy}-1}{|y|}e^{-iuy}\,dy=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{0}^{\infty}\frac{\cos(x+u)y-\cos uy}{y}\,dy$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\lim_{\delta\to 0}\int\limits_{\delta}^{\infty}\frac{\cos(x+u)y-\cos uy}{y}\,dy$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\lim_{\delta\to 0}\left(\int\limits_{\delta|x+u|}^{\infty}\frac{\cos v}{v}\,dv-\int\limits_{\delta|u|}^{\infty}\frac{\cos v}{v}\,dv\right)$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\lim_{\delta\to 0}\int\limits_{\delta|x+u|}^{\delta|u|}\frac{\cos v}{v}\,dv$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\lim_{\delta\to 0}\int\limits_{\delta|x+u|}^{\delta|u|}\frac{dv}{v}=\sqrt{\left(\frac{2}{\pi}\right)}\log\left|\frac{u}{x+u}\right|.$$

Hence Parseval's formula gives

$$\int\limits_{-\infty}^{\infty}F(y)\frac{e^{-ixy}-1}{|y|}\,dy=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}f(u)\log\left|\frac{u}{x-u}\right|\,du,$$

and (5.2.1) follows. The relation between F and G, and so between f and g, is skew-reciprocal, so that (5.2.2) also follows.

5.3. THEOREM 91.† *Let $f(x)$ belong to $L^2(-\infty,\infty)$. Then the formula*

$$g(x)=\frac{1}{\pi}\int\limits_{\to 0}^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.3.1}$$

defines almost everywhere a function $g(x)$, also of $L^2(-\infty,\infty)$. The reciprocal formula

$$f(x)=-\frac{1}{\pi}\int\limits_{\to 0}^{\infty}\frac{g(x+t)-g(x-t)}{t}\,dt \tag{5.3.2}$$

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† The analogue for series is due to Plessner (1). See also Hardy (14).

also holds almost everywhere; and

$$\int_{-\infty}^{\infty} \{f(x)\}^2\,dx = \int_{-\infty}^{\infty} \{g(x)\}^2\,dx. \tag{5.3.3}$$

The functions $g(x)$ of Theorems 90 and 91 are equivalent.

The integrals (5.1.2) defining $a(t)$ and $b(t)$ exist in the mean-square sense, and

$$a(t)-ib(t) = \sqrt{\left(\frac{2}{\pi}\right)}F(-t).$$

Let $H(t) = e^{izt}$ $(t > 0)$, 0 $(t < 0)$. Then

$$h(u) = \frac{1}{\sqrt{(2\pi)}}\int_{0}^{\infty} e^{izt-izu}\,du = \frac{1}{i\sqrt{(2\pi)}(u-z)}.$$

Hence Parseval's formula, in the form

$$\int_{-\infty}^{\infty} F(-t)H(t)\,dt = \int_{-\infty}^{\infty} f(t)h(t)\,dt,$$

applied to (5.1.4) gives

$$\Phi(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt \quad (\mathbf{I}(z) > 0). \tag{5.3.4}$$

Taking real and imaginary parts separately, we obtain

$$U(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2+y^2}\,dt, \tag{5.3.5}$$

$$V(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{(t-x)^2+y^2}f(t)\,dt. \tag{5.3.6}$$

Define g and G as in § 5.2, the integrals being now mean-square. Then we also have

$$\begin{aligned}\Phi(z) &= \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} F(-t)\,e^{izt}\,dt = -i\sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} G(-t)\,e^{izt}\,dt \\ &= -i\sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty} G(-t)H(t)\,dt = -i\sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty} g(t)h(t)\,dt \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t-z}\,dt.\end{aligned} \tag{5.3.7}$$

Hence

$$U(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{(t-x)^2+y^2} g(t)\,dt, \tag{5.3.8}$$

$$V(x,y) = -\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{(t-x)^2+y^2}\,dt. \tag{5.3.9}$$

By the theory of Cauchy's singular integral, § 1.17, $U(x,y) \to f(x)$ as $y \to 0$ for almost all values of x, and $V(x,y) \to -g(x)$ for almost all values of x. We now use the following theorem.

THEOREM 92. *Let $f(x)$ be any function such that $f(x)$ belongs to $L(0,1)$, and $x^{-1}f(x)$ to $L(1,\infty)$. Let $V(x,y)$ be defined by* (5.3.6). *Then*

$$\lim_{y\to 0}\left\{V(x,y)+\frac{1}{\pi}\int_{y}^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt\right\} = 0 \tag{5.3.10}$$

for almost all values of x.

We know that

$$\omega(y) = \int_0^y |f(x+t)-f(x-t)|\,dt = o(y)$$

for almost all values of x. Let x be a point where this holds. We have

$$V(x,y)+\frac{1}{\pi}\int_y^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt$$

$$= -\frac{1}{\pi}\int_0^y \frac{t}{t^2+y^2}\{f(x+t)-f(x-t)\}\,dt +$$

$$+\frac{y^2}{\pi}\int_y^1 \frac{f(x+t)-f(x-t)}{(t^2+y^2)t}\,dt + \frac{y^2}{\pi}\int_1^{\infty}\frac{f(x+t)-f(x-t)}{(t^2+y^2)t}\,dt$$

$$= J_1+J_2+J_3$$

say. As $y \to 0$,

$$|J_1| \leqslant \frac{1}{2\pi y}\int_0^y |f(x+t)-f(x-t)|\,dt = o(1),$$

$$|J_2| \leqslant \frac{y^2}{\pi}\int_y^1 \frac{|f(x+t)-f(x-t)|}{(t^2+y^2)t}\,dt$$

$$= \frac{y^2}{\pi}\left[\frac{\omega(t)}{(t^2+y^2)t}\right]_y^1 + \frac{y^2}{\pi}\int_y^1 \frac{3t^2+y^2}{(t^2+y^2)^2t^2}\,\omega(t)\,dt$$

$$\leqslant \frac{y^2}{\pi}\frac{\omega(1)}{1+y^2}+o\left\{y^2\int\limits_{y}^{1}\frac{3t^2+y^2}{(t^2+y^2)^2t}\,dt\right\}$$

$$= O(y^2)+o\left\{\int\limits_{1}^{1/y}\frac{3u^2+1}{(u^2+1)^2u}\,du\right\} = o(1),$$

and plainly $J_3 = o(1)$. Hence the result.

Since $V(x,y) \to -g(x)$ almost everywhere, (5.3.1) follows from (5.3.10). The relation between f and g being skew-reciprocal, (5.3.2) also follows; and (5.3.3) holds as before.

5.4. In this section we shall show that the same set of formulae may be obtained from a different source. We can take an analytic function $\Phi(z)$ satisfying certain conditions as the original function.

THEOREM 93. *Let* $\Phi(z)$ *be an analytic function, regular for* $y > 0$, *and let*

$$\int\limits_{-\infty}^{\infty} |\Phi(x+iy)|^2\,dx$$

exist for every positive y, *and be bounded. Then, as* $y \to 0$, $\Phi(x+iy)$ *converges in mean to a function* $\Phi(x)$, *and also* $\Phi(x+iy) \to \Phi(x)$ *for almost all* x. *For* $y > 0$

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\Phi(u)}{u-z}\,du \quad (u \textit{ real}).$$

If $\Phi(z) = U(x,y)+iV(x,y)$, $\Phi(x) = f(x)-ig(x)$, *the functions* U, V, f, *and* g *are connected by the formulae of the previous section, and in particular* f *and* g *are conjugate.*

We first prove the following

LEMMA. *Let* $\Phi(z)$ *be analytic, and let*

$$\int\limits_{-\infty}^{\infty} |\Phi(x+iy)|^p\,dx \quad (p > 1)$$

exist and be bounded for $y_1 \leqslant y \leqslant y_2$. *Then, as* $x \to \pm\infty$, $\Phi(x+iy) \to 0$, *uniformly for* $y_1+\delta \leqslant y \leqslant y_2-\delta$.

Let $y_1+\delta \leqslant y \leqslant y_2-\delta$. Then, if $0 < \rho \leqslant \delta$,

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{|w-z|=\rho}\frac{\Phi(w)}{w-z}\,dw = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\Phi(z+\rho e^{i\phi})\,d\phi.$$

Hence
$$\tfrac{1}{2}\delta^2\Phi(z) = \frac{1}{2\pi}\int_0^\delta \rho\, d\rho \int_0^{2\pi} \Phi(z+\rho e^{i\phi})\, d\phi,$$

$$\tfrac{1}{2}\delta^2|\Phi(z)| \leqslant \frac{1}{2\pi}\left\{\int_0^\delta\int_0^{2\pi} |\Phi|^p \rho\, d\rho d\phi\right\}^{1/p}\left\{\int_0^\delta\int_0^{2\pi} \rho\, d\rho d\phi\right\}^{1-1/p}$$

$$\leqslant K(\delta)\left\{\int_{y_1}^{y_2} dv \int_{x-\delta}^{x+\delta} |\Phi(u+iv)|^p\, du\right\}^{1/p}$$

Now
$$\int_{x-\delta}^{x+\delta} |\Phi(u+iv)|^p\, du$$
is bounded for $y_1 \leqslant v \leqslant y_2$, and tends to 0 as $x \to \infty$, for every v. Hence the right-hand side tends to 0, and the result follows.

To prove the theorem, let
$$\phi_a(t,y) = \frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a} \Phi(z)e^{-itx}\, dx.$$

For each y this converges in mean, to $\phi(t,y)$ say, as $a \to \infty$. Consider, however, the integral
$$\int \Phi(z)e^{-itz}\, dz$$
taken round the rectangle with corners at $\pm a+iy_1$, $\pm a+iy_2$, where $0 < y_1 < y_2$. The integral along the right-hand side is
$$\int_{y_1}^{y_2} \Phi(a+iy)e^{-it(a+iy)}i\, dy = ie^{-ita}\int_{y_1}^{y_2} \Phi(a+iy)e^{ty}\, dy,$$
and, by the lemma, this tends to 0 as $a \to \infty$, for fixed y_1 and y_2. Similarly, the integral along the left-hand side tends to 0. Hence, as $a \to \infty$,
$$\int_{-a}^{a} \Phi(x+iy_1)e^{-it(x+iy_1)}\, dx - \int_{-a}^{a} \Phi(x+iy_2)e^{-it(x+iy_2)}\, dx \to 0,$$
i.e.
$$e^{ty_1}\phi_a(t,y_1) - e^{ty_2}\phi_a(t,y_2) \to 0.$$
Hence the mean-square limit of this sequence over any finite interval is also 0, i.e.
$$e^{ty_1}\phi(t,y_1) = e^{ty_2}\phi(t,y_2)$$
for almost all t. We may therefore write
$$\phi(t,y) = e^{-ty}\phi(t),$$
$\phi(t)$ being independent of y (e.g. by putting $\phi(t) = e^{t}\phi(t,1)$).

Also, by Parseval's theorem,

$$\int_{-\infty}^{\infty} |\phi(t)|^2 e^{-2ty}\, dt = \int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx.$$

Since this is bounded as $y \to \infty$, we must have $\phi(t) \equiv 0$ for $t < 0$; for

$$\int_{-\infty}^{-\delta} |\phi(t)|^2\, dt \leqslant e^{-2\delta y} \int_{-\infty}^{-\delta} |\phi(t)|^2 e^{-2ty}\, dt < Ke^{-2\delta y} \to 0,$$

so that
$$\int_{-\infty}^{-\delta} |\phi(t)|^2\, dt = 0.$$

Since it is also bounded as $y \to 0$, $\phi(t)$ belongs to $L^2(0,\infty)$.

Also, $\phi(t)(e^{-ty_1}-e^{-ty_2})$ is the transform of $\Phi(x+iy_1)-\Phi(x+iy_2)$. Hence

$$\int_{-\infty}^{\infty} |\Phi(x+iy_1)-\Phi(x+iy_2)|^2\, dx = \int_{0}^{\infty} |\phi(t)|^2 (e^{-ty_1}-e^{-ty_2})^2\, dt,$$

which tends to 0 as $y_1 \to 0$, $y_2 \to 0$. Hence $\Phi(x+iy)$ converges in mean as $y \to 0$, to $\Phi(x)$ say.

Next, if $y > 0$, $$\Phi(z) = \frac{1}{2\pi i} \int \frac{\Phi(w)}{w-z}\, dw,$$

the integral being taken round the rectangle $\pm a+iv_1$, $\pm a+iv_2$, where $a > |x|$ and $v_1 < y < v_2$. As before, the integrals along the right- and left-hand sides tend to 0 as $a \to \infty$, and we obtain

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u+iv_1)}{u+iv_1-z}\, du - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u+iv_2)}{u+iv_2-z}\, du.$$

But

$$\left| \int_{-\infty}^{\infty} \frac{\Phi(u+iv_2)}{u+iv_2-z}\, du \right|^2 \leqslant \int_{-\infty}^{\infty} |\Phi(u+iv_2)|^2\, du \int_{-\infty}^{\infty} \frac{du}{(u-x)^2+(v_2-y)^2}$$
$$< \frac{K}{v_2-y},$$

which tends to 0 as $v_2 \to \infty$. Hence

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u+iv_1)}{u+iv_1-z}\, du, \tag{5.4.1}$$

and, making $v_1 \to 0$,

$$\Phi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi(u)}{u-z}\,du. \tag{5.4.2}$$

The integral with z replaced by $\bar{z}$ is zero. Putting $\Phi(u) = f(u) - ig(u)$, $\Phi(z) = U(x,y)+iV(x,y)$, we obtain the formulae of the previous sections; and it follows from the theory of Cauchy's singular integral, and (5.3.5) and (5.3.9), that $\Phi(z) \to \Phi(x)$ for almost all x.

THEOREM 94. *If $\psi(z)$ is regular and bounded for $y > 0$, then $\psi(z)$ tends to a limit as $y \to 0$ for almost all x.*

For $\psi(z)/(z+i)$ satisfies the conditions of the above theorem, and so tends to a limit almost everywhere.

Notice also that, in the above theorem, $\phi(t)$ is the transform of $\Phi(x)$; for, if $\chi(t)$ is the transform of $\Phi(x)$, as $y \to 0$,

$$\int_{-\infty}^{\infty} |\chi(t)-\phi(t)e^{-tv}|^2\,dt = \int_{-\infty}^{\infty} |\Phi(x)-\Phi(x+iy)|^2\,dx \to 0,$$

$$\int_{-\infty}^{\infty} |\chi(t)-\phi(t)|^2\,dt = 0.$$

Hence $\chi(t) \equiv \phi(t)$.

5.5. We also deduce

THEOREM 95. *Alternative necessary and sufficient conditions that a complex $\Phi(x)$ of $L^2(-\infty,\infty)$ should be the limit as $z \to x$ of an analytic $\Phi(z)$ such that*

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\,dx < K$$

are

(i) $\Phi(x) = f(x)-ig(x)$, *where f and g are conjugate functions of the class L^2;*

(ii) $\phi(x)$, *the transform of $\Phi(x)$, is null for $x < 0$.*

The necessity and sufficiency of (i) follows at once from the above theorems.

The necessity of (ii) has been proved in the course of the previous proof. Conversely, let $\phi(x) \equiv 0$ for $x < 0$. Let Φ be its transform. Then

$$\Phi(u) = \underset{a\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{(2\pi)}}\int_{0}^{a} e^{ixu}\phi(x)\,dx.$$

Let $$\Phi(u+iv) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty e^{ix(u+iv)}\phi(x)\,dx \quad (v > 0).$$

Then $\Phi(u+iv)$ is analytic for $v > 0$, and

$$\int_{-\infty}^{\infty} |\Phi(u+iv)|^2\,du = \int_0^\infty e^{-2xv}|\phi(x)|^2\,dx \leqslant \int_0^\infty |\phi(x)|^2\,dx.$$

Hence, by the above theorem, $\Phi(u+iv)$ converges in mean, and also almost everywhere, to $\Psi(u)$ say; and

$$\begin{aligned}\int_0^U \Phi(u)\,du &= \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \frac{e^{ixU}-1}{ix}\phi(x)\,dx\\ &= \lim_{v\to 0}\frac{1}{\sqrt{(2\pi)}}\int_0^\infty \frac{e^{ix(U+iv)}-1}{ix}\phi(x)\,dx\\ &= \lim_{v\to 0}\int_0^U \Phi(u+iv)\,du = \int_0^U \Psi(u)\,du.\end{aligned}$$

Hence $$\Psi(u) \equiv \Phi(u).$$

The result also follows from the transform formulae; for, if Φ satisfies the given conditions, Φ, f, and g are related as in § 5.1, and

$$\begin{aligned}\phi(x) &= \text{l.i.m.}\frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a} \{f(u)-ig(u)\}e^{-ixu}\,du\\ &= F(-x)-iG(-x) = 0 \quad (x < 0)\end{aligned}$$

by (5.1.8). Conversely, let $\phi(x) \equiv 0$ for $x < 0$. Let $\Phi(u) = f(u)-ig(u)$, let $a(x)$ and $b(x)$ be defined as before in terms of f, and similarly $\alpha(x)$ and $\beta(x)$ in terms of g. Then

$$a(x)+ib(x)-i\{\alpha(x)+i\beta(x)\} = 0 \quad (x < 0),$$

i.e. $$a(x) = -\beta(x), \quad b(x) = \alpha(x) \qquad (x < 0).$$

Hence g is the conjugate of f, and the sufficiency of the condition follows from condition (i).

5.6. THEOREM 96. *A necessary and sufficient condition that $\Phi(x)$ should be the limit as $y \to 0$ of an analytic $\Phi(z)$ such that*

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\,dx = O(e^{2ky})$$

is that $\phi(x) \equiv 0$ for $x < -k$.

If k is the least number such that $\phi(x) \equiv 0$ for $x < -k$, then

$$\lim_{y\to\infty} \frac{1}{y} \log \int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx = 2k.$$

Suppose that $\Phi(z)$ satisfies the given conditions. Let

$$\Phi(z) = e^{-ikz}\Psi(z).$$

Then $$\int_{-\infty}^{\infty} |\Psi(x+iy)|^2\, dx = e^{-2ky} \int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx = O(1).$$

Hence $\Psi(z) \to \Psi(x)$ almost everywhere, and, if $\psi(x)$ is the transform of $\Psi(x)$, $\psi(x) \equiv 0$ for $x < 0$. Now

$$\phi(x) = \text{l.i.m.} \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} \Phi(u)e^{-ixu}\, du$$

$$= \text{l.i.m.} \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} \Psi(u)e^{-i(x+k)u}\, du = \psi(x+k).$$

Hence $\phi(x) \equiv 0$ for $x < -k$; and in view of the above theorem the argument is reversible. This proves the first part.

Again, since $\Phi(x+iy)$ is the transform of $e^{-uy}\phi(u)$,

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx = \int_{-\infty}^{\infty} e^{-2uy}|\phi(u)|^2\, du = \int_{-k}^{\infty} e^{-2uy}|\phi(u)|^2\, du.$$

This is $$\leqslant e^{2ky} \int_{-k}^{\infty} |\phi(u)|^2\, du;$$

and, if $$\omega(u) = \int_{-k}^{u} \phi|(u)|^2\, du,$$

it equals $$2y \int_{-k}^{\infty} e^{-2uy}\omega(u)\, du \geqslant 2y\omega(-k+\delta) \int_{-k+\delta}^{\infty} e^{-2uy}\, du$$
$$= \omega(\delta-k)e^{2(k-\delta)y}.$$

Hence the second part.

5.7. For a function having a mean value in a finite strip the corresponding theorem is as follows.

THEOREM 97. *Let $\Phi(z)$ be an analytic function, regular for*

$$y_1 < y < y_2,$$

and such that $$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2\, dx$$

exists and is bounded for $y_1 < y < y_2$.

Then the boundary functions $\Phi(x+iy_1)$ *and* $\Phi(x+iy_2)$ *exist as mean-square limits, and also almost everywhere as ordinary limits of* $\Phi(x+iy)$. *For* $y_1 < y < y_2$,

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Phi(u+iy_1)}{u+iy_1-z}\,du - \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Phi(u+iy_2)}{u+iy_2-z}\,du.$$

The transform of $\Phi(x+iy)$ *is of the form* $e^{-ty}\phi(t)$, *where* $e^{-ty}\phi(t)$ *belongs to* L^2 *for* $y_1 \leqslant y \leqslant y_2$.

This is an obvious consequence of the above analysis, except perhaps for the existence of the limit of $\Phi(x+iy)$ almost everywhere as $y \to y_1$ or y_2. However, the previous analysis shows that

$$\int\limits_{-\infty}^{\infty} \frac{\Phi(u+iy_1)}{u+iy_1-z}\,du$$

tends to a limit almost everywhere as $y \to y_1$ from above; and

$$\int\limits_{-\infty}^{\infty} \frac{\Phi(u+iy_2)}{u+iy_2-z}\,du$$

is regular for all $y < y_2$, and so tends to a limit everywhere as $y \to y_1$. Similarly for the case $y \to y_2$.

5.8. THEOREM 98. *Let* $f(x)$ *belong to* $L^2(-\infty,\infty)$. *Then*

$$f(x) = f_+(x)+f_-(x),$$

where $f_+(x)$ *belongs to* $L^2(-\infty,\infty)$, *and is the mean-square limit of an analytic function* $f_+(z)$, *regular for* $\mathrm{I}(z) > 0$; *and similarly* $f_-(x)$ *is the mean-square limit of* $f_-(z)$, *regular for* $\mathrm{I}(z) < 0$.

Let $F(x)$ be the transform of $f(x)$, and

$$f_+(z) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{0}^{\infty} F(u)e^{-izu}\,du, \qquad f_-(z) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{0} F(u)e^{-izu}\,du.$$

Plainly $f_+(z)$ and $f_-(z)$ are regular for $y > 0$, $y < 0$, respectively. The rest of the theorem follows as in § 5.4.

THEOREM 99. *Let* $f(x)$ *belong to* $L^2(0,\infty)$. *Then*

$$f(x) = f_{(+)}(x)+f_{(-)}(x),$$

where $f_{(+)}(x)$ *belongs to* $L^2(0,\infty)$, *and is the mean-square limit as* $\arg z \to +0$ *of an analytic function* $f_{(+)}(z)$, *regular for* $\arg z > 0$; *and similarly* $f_{(-)}(x)$ *is the mean-square limit of* $f_{(-)}(z)$, *regular for* $\arg z < 0$.

This may be deduced from the previous theorem by putting z (of the present theorem) $= e^{\zeta}$; or deduced directly from Mellin transforms. In fact

$$f_{(+)}(z) = \frac{1}{2\pi i}\int\limits_{\frac12-i\infty}^{\frac12} \mathfrak{F}(s)z^{-s}\,ds, \qquad f_{(-)}(z) = \frac{1}{2\pi i}\int\limits_{\frac12}^{\frac12+i\infty} \mathfrak{F}(s)z^{-s}\,ds.$$

5.9. THEOREM 100†. *If $f(x)$ belongs to $L(-\infty,\infty)$, then*

$$\int\limits_{\to 0}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt$$

exists for almost all values of x.

We may suppose without loss of generality that $f(x) \geqslant 0$. Define $U(x,y)$ and $V(x,y)$ by (5.3.5), (5.3.6), and let

$$\Phi(z) = U(x,y)+iV(x,y) = \frac{1}{i\pi}\int\limits_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt \quad (y > 0).$$

From its definition it is clear that $U(x,y) \geqslant 0$

Let $$\Psi(z) = e^{-\Phi(z)} = e^{-U(x,y)-iV(x,y)}.$$

Since $U(x,y) \geqslant 0$, $|\Psi(z)| \leqslant 1$. Hence, as $y \to 0$, $\Psi(z)$ tends to a finite limit for almost all x (Theorem 94); and this limit can be 0 in a set of measure 0 only, since $U(x,y)$ tends to the finite limit $f(x)$ almost everywhere. Hence $\Psi(z)$ tends to a finite non-zero limit almost everywhere. Hence $\Phi(z)$ tends to a finite limit almost everywhere. Hence $V(x,y)$ tends to a finite limit almost everywhere. The result then follows from Theorem 92.

5.10. Hilbert transforms of the class L^p.

THEOREM 101. *Let $f(x)$ belong to $L^p(-\infty,\infty)$, where $p > 1$. Then the formula*

$$g(x) = \frac{1}{\pi}\int\limits_{\to 0}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.10.1}$$

defines almost everywhere a function $g(x)$, also belonging to $L^p(-\infty,\infty)$. The reciprocal formula

$$f(x) = -\frac{1}{\pi}\int\limits_{\to 0}^{\infty} \frac{g(x+t)-g(x-t)}{t}\,dt \tag{5.10.2}$$

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† Plessner (1). I believe that this version of the argument is due to Littlewood.

also holds almost everywhere; and

$$\int_{-\infty}^{\infty} |g(x)|^p\,dx \leqslant M_p^p \int_{-\infty}^{\infty} |f(x)|^p\,dx, \tag{5.10.3}$$

where M_p depends on p only.

This is M. Riesz's extension† of Theorem 91. There are three cases.

(i) Let p be an even integer. Let

$$\Phi_a(z) = \frac{1}{i\pi}\int_{-a}^{a} \frac{f(t)}{t-z}\,dt = U_a(x,y)+iV_a(x,y) \quad (y>0).$$

Consider the integral $\int \{\Phi_a(z)\}^p\,dz$

taken along the straight line from $-R+iy$ to $R+iy$, and round the semicircle above it. For a fixed a, $\Phi_a(z) = O(1/|z|)$ as $|z|\to\infty$. Hence, making $R\to\infty$, we obtain

$$\int_{-\infty}^{\infty} \{\Phi_a(x+iy)\}^p\,dx = 0,$$

i.e.

$$\int_{-\infty}^{\infty} (U_a+iV_a)^p\,dx = 0.$$

Expanding the integrand by the binomial theorem, and taking the real part,

$$\int_{-\infty}^{\infty} \left\{V_a^p-\binom{p}{2}V_a^{p-2}U_a^2+\binom{p}{4}V_a^{p-4}U_a^4-\ldots\pm U_a^p\right\}dx = 0.$$

Hence
$$\int_{-\infty}^{\infty} V_a^p\,dx \leqslant \binom{p}{2}\int_{-\infty}^{\infty} V_a^{p-2}U_a^2\,dx+\ldots+\int_{-\infty}^{\infty} U_a^p\,dx.$$

Now
$$\int_{-\infty}^{\infty} V_a^{p-2k}U_a^{2k}\,dx \leqslant \left(\int_{-\infty}^{\infty} V_a^p\,dx\right)^{(p-2k)/p}\left(\int_{-\infty}^{\infty} U_a^p\,dx\right)^{2k/p}$$

Writing
$$X^p = \left(\int_{-\infty}^{\infty} V_a^p\,dx\right)\Big/\left(\int_{-\infty}^{\infty} U_a^p\,dx\right),$$

it follows that

$$X^p \leqslant \binom{p}{2}X^{p-2}+\binom{p}{4}X^{p-4}+\ldots+1.$$

Hence X does not exceed the greatest positive root of the equation

$$X^p-\binom{p}{2}X^{p-2}-\ldots-1 = 0,$$

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† M. Riesz (1), (2). For another method see Titchmarsh (7).

and so
$$X \leqslant M_p,$$
where M_p depends on p only; i.e.
$$\int_{-\infty}^{\infty} V_a^p\,dx \leqslant M_p^p \int_{-\infty}^{\infty} U_a^p\,dx.$$
Now
$$|U_a(x,y)| = \frac{y}{\pi}\left|\int_{-a}^{a} \frac{f(t)}{(t-x)^2+y^2}\,dt\right| \leqslant \frac{y}{\pi}\int_{-a}^{a} \frac{|f(t)|}{(t-x)^2+y^2}\,dt,$$
$$|U_a(x,y)|^p \leqslant \frac{y^p}{\pi^p}\int_{-\infty}^{\infty} \frac{|f(t)|^p}{(t-x)^2+y^2}\,dt\left\{\int_{-\infty}^{\infty} \frac{dt}{(t-x)^2+y^2}\right\}^{p-1}$$
$$= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|f(t)|^p}{(t-x)^2+y^2}\,dt,$$
$$\int_{-\infty}^{\infty} |U_a|^p\,dx \leqslant \frac{y}{\pi}\int_{-\infty}^{\infty} |f(t)|^p\,dt \int_{-\infty}^{\infty} \frac{dx}{(t-x)^2+y^2}$$
$$= \int_{-\infty}^{\infty} |f(t)|^p\,dt. \tag{5.10.4}$$
Hence
$$\int_{-\infty}^{\infty} \{V_a(x,y)\}^p\,dx \leqslant M_p^p \int_{-\infty}^{\infty} \{f(t)\}^p\,dt.$$
Making $a \to \infty$,
$$V_a(x,y) \to V(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{(t-x)^2+y^2}f(t)\,dt, \tag{5.10.5}$$
and, making $y \to 0$, $V(x,y) \to -g(x)$ almost everywhere, by Theorems 92 and 100. It therefore follows from (5.10.5) and Fatou's theorem that (5.10.3) holds. (See Titchmarsh, *Theory of Functions*, § 10.81.)

(ii) Suppose next that p is not an integer. We may suppose without loss of generality that $f(t) \geqslant 0$. Then $U(x,y) > 0$, and $U_a(x,y) > 0$ for $y > 0$, $a > a_0$.

Some care is now needed in the definition of pth powers. Let
$$(U+iV)^p = e^{\frac{1}{2}p\log(U^2+V^2)+ip\arctan(V/U)},$$
where $-\frac{1}{2}\pi < \arctan(V/U) < \frac{1}{2}\pi$ for $U > 0$. Making $U \to 0$, we obtain
$$(iV)^p = |V|^p e^{\frac{1}{2}ip\pi} \quad (V > 0), \qquad |V|^p e^{-\frac{1}{2}ip\pi} \quad (V < 0).$$

With these definitions we have

$$|(U+iV)^p-(iV)^p| = p\left|\int\limits_{iV}^{U+iV} z^{p-1}\,dz\right|$$

$$\leqslant pU(U^2+V^2)^{\frac{1}{2}p-\frac{1}{2}} \leqslant p\,2^{\frac{1}{2}(p-1)}(U^p+U|V|^{p-1}).$$

Applying this to U_a, V_a, we obtain

$$\left|\int\limits_{-\infty}^{\infty} \{(U_a+iV_a)^p-(iV_a)^p\}\,dx\right| \leqslant K_p\left\{\int\limits_{-\infty}^{\infty} U_a^p\,dx + \int\limits_{-\infty}^{\infty} U_a|V_a|^{p-1}\,dx\right\}.$$

But, as before, $$\int\limits_{-\infty}^{\infty} \{U_a+iV_a\}^p\,dx = 0,$$

and $$\left|\int\limits_{-\infty}^{\infty} (iV_a)^p\,dx\right| \geqslant \left|\mathbf{R}\int\limits_{-\infty}^{\infty} (iV_a)^p\,dx\right| = |\cos\tfrac{1}{2}p\pi|\int\limits_{-\infty}^{\infty} |V_a|^p\,dx.$$

Hence

$$|\cos\tfrac{1}{2}p\pi|\int\limits_{-\infty}^{\infty} |V_a|^p\,dx \leqslant K_p\int\limits_{-\infty}^{\infty} U_a^p\,dx + K_p\int\limits_{-\infty}^{\infty} U_a|V_a|^{p-1}\,dx,$$

and the proof of (5.10.1) and (5.10.3) can now be completed as in the previous case.

The above proof fails if p is an odd integer. Leaving this case aside for the moment, we next prove (5.10.2) in the above cases.

We have

$$|U(x,y)-f(x)|^p \leqslant \frac{y^p}{\pi^p}\left|\int\limits_{-\infty}^{\infty} \frac{f(t+x)-f(x)}{t^2+y^2}\,dt\right|^p$$

$$< \frac{y}{\pi}\int\limits_{-\infty}^{\infty} \frac{|f(t+x)-f(x)|^p}{t^2+y^2}\,dt,$$

$$\int\limits_{-\infty}^{\infty} |U(x,y)-f(x)|^p\,dx < \frac{y}{\pi}\int\limits_{-\infty}^{\infty} \frac{dt}{t^2+y^2}\int\limits_{-\infty}^{\infty} |f(t+x)-f(x)|^p\,dx.$$

The inner integral (see Titchmarsh, *Theory of Functions*, p. 397, exs. 17–19) is bounded for all t, and tends to 0 with t; hence the right-hand side is less than

$$K_p y\int\limits_{\delta}^{\infty} \frac{dt}{t^2+y^2}+\epsilon(\delta)y\int\limits_{0}^{\delta} \frac{dt}{t^2+y^2} < K_p y\int\limits_{\delta}^{\infty} \frac{dt}{t^2}+\epsilon(\delta)y\int\limits_{0}^{\infty} \frac{dt}{t^2+y^2}$$

$$< K_p y/\delta+\tfrac{1}{2}\pi\epsilon(\delta),$$

which tends to 0 by choosing first δ and then y. Hence for any p

$$\lim_{y\to 0}\int_{-\infty}^{\infty} |U(x,y)-f(x)|^p\,dx = 0.$$

Also, by (5.10.4),

$$\int_{-\infty}^{\infty} |U(x,y)-U_a(x,y)|^p\,dx \leqslant \left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)|f(t)|^p\,dt \to 0$$

as $a\to\infty$, uniformly with respect to y. Hence

$$\int_{-\infty}^{\infty} |U_a(x,y)-f(x)|^p\,dx \to 0 \tag{5.10.6}$$

as $a\to\infty$, $y\to 0$, in any manner.

Again, by the calculus of residues,

$$\frac{1}{\pi i}P\int_{y=\eta} \frac{\Phi_a(z)}{z-\xi-i\eta}\,dz = \Phi_a(\xi+i\eta) \quad (\eta>0),$$

and, taking imaginary parts,

$$\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{U_a(x,\eta)}{x-\xi}\,dx = -V_a(\xi,\eta).$$

Hence the Hilbert transform of $U_a(x,y)$ is $-V_a(x,y)$, and it follows from (5.10.3) that, for the values of p already dealt with,

$$\int_{-\infty}^{\infty} |V_a(x,y)+g(x)|^p < K_p \int_{-\infty}^{\infty} |U_a(x,y)-f(x)|^p\,dx. \tag{5.10.7}$$

Combining (5.10.6) and (5.10.7), it follows that

$$\int_{-\infty}^{\infty} |\Phi_a(z)-\{f(x)-ig(x)\}|^p\,dx \to 0$$

as $y\to 0$, $a\to\infty$, in any manner.

Now by the calculus of residues

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi_a(z)}{z-\xi-i\eta}\,dz = \Phi_a(\xi+i\eta) \quad (y<\eta).$$

Making $a\to\infty$, $y\to 0$, it follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)-ig(x)}{x-\xi-i\eta}\,dx = \Phi(\xi+i\eta),$$

and hence
$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{-ig(x)}{x-\xi-i\eta}\,dx = \tfrac{1}{2}\Phi(\xi+i\eta).$$

Taking real parts,
$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}g(x)\,dx = -U(\xi,\eta).$$

Making $\eta \to 0$, the left-hand side tends almost everywhere to the Hilbert transform of $g(x)$, by Theorem 92; and the right-hand side (Cauchy's singular integral) tends almost everywhere to $-f(x)$. This proves (5.10.2).

(iii) To prove the case where p is an odd integer, we shall prove that if the theorem holds for any p it also holds for $2p$. Since it holds when p is half an odd integer, it will follow that it holds when p is an odd integer.

Applying the calculus of residues as before, but now to $\{\Phi_a(z)\}^2$, we obtain
$$\frac{1}{\pi i}P\int\limits_{-\infty}^{\infty}\frac{\{\Phi_a(x+iy)\}^2}{x-\xi}\,dx = \{\Phi_a(\xi+iy)\}^2 \quad (y>0),$$
i.e.
$$\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{U_a^2-V_a^2+2iU_aV_a}{x-\xi}\,dx = U_a^2(\xi,y)-V_a^2(\xi,y)+2iU_a(\xi,y)V_a(\xi,y).$$

Taking imaginary parts, it follows that the Hilbert transform of $U_a^2-V_a^2$ is $-2U_aV_a$. Let $\psi(x)$ be the transform of U_a^2, and $\chi(x)$ that of V_a^2. Then
$$\psi(x)-\chi(x) = -2U_aV_a.$$

Hence
$$|\chi(x)|^p \leqslant 2^p|\psi(x)|^p+2^{2p}|U_aV_a|^p,$$
$$\int\limits_{-\infty}^{\infty}|\chi(x)|^p\,dx \leqslant 2^p\int\limits_{-\infty}^{\infty}|\psi(x)|^p\,dx+2^{2p}\int\limits_{-\infty}^{\infty}|U_aV_a|^p\,dx.$$

Now
$$\int\limits_{-\infty}^{\infty}|U_aV_a|^p\,dx \leqslant \left(\int\limits_{-\infty}^{\infty}|U_a|^{2p}\,dx\int\limits_{-\infty}^{\infty}|V_a|^{2p}\,dx\right)^{\frac{1}{2}},$$
and, by the fundamental inequality (5.10.3) (for p),
$$\int\limits_{-\infty}^{\infty}|\psi(x)|^p\,dx < K_p\int\limits_{-\infty}^{\infty}|U_a|^{2p}\,dx,$$
$$\int\limits_{-\infty}^{\infty}|V_a|^{2p}\,dx < K_p\int\limits_{-\infty}^{\infty}|\chi(x)|^p\,dx.$$

Altogether

$$\int_{-\infty}^{\infty} |V_a|^{2p}\,dx < K_p \int_{-\infty}^{\infty} |U_a|^{2p}\,dx + K_p\left(\int_{-\infty}^{\infty} |U_a|^{2p}\,dx \int_{-\infty}^{\infty} |V_a|^{2p}\,dx\right)^{\frac{1}{2}}$$

The result for $2p$ now follows as in the previous cases. This completes the proof.

5.11. THEOREM 102.† *Let $f(x)$ and $g(x)$ be Hilbert transforms of the class L^p, and $h(x)$ and $k(x)$ Hilbert transforms of the class $L^{p'}$, where $p' = p/(p-1)$. Then*

$$\int_{-\infty}^{\infty} f(x)h(x)\,dx = \int_{-\infty}^{\infty} g(x)k(x)\,dx. \tag{5.11.1}$$

If $p = 2$, $p' = 2$, and (5.3.3) gives

$$\int_{-\infty}^{\infty} \{f(x)+h(x)\}^2\,dx = \int_{-\infty}^{\infty} \{g(x)+k(x)\}^2\,dx, \tag{5.11.2}$$

and the result follows in the usual way.

In the general case, define $U_a(x,y)$, $V_a(x,y)$ as before, and let $P_b(x,y)$, $Q_b(x,y)$ be the corresponding functions for h and k. We have seen that the Hilbert transform of U_a is $-V_a$, and similarly that of P_b is $-Q_b$. Since these functions belong to L^2,

$$\int_{-\infty}^{\infty} U_a(x,y)P_b(x,y')\,dx = \int_{-\infty}^{\infty} V_a(x,y)Q_b(x,y')\,dx. \tag{5.11.3}$$

Making $a \to \infty$, $y \to 0$, $b \to \infty$, $y' \to 0$, U_a and V_a converge in mean to f and $-g$, with exponent p; and P_b and Q_b converge in mean to h and $-k$, with exponent p'. Hence the result.

EXAMPLE. Let $h(x) = 1/(x-a)$ $(|x-a| > \delta)$, 0 $(|x-a| \leqslant \delta)$. Then

$$k(x) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{h(u)}{u-x}\,du = \frac{1}{\pi}P\left\{\int_{-\infty}^{a-\delta} \frac{du}{(u-a)(u-x)} + \int_{a+\delta}^{\infty} \frac{du}{(u-a)(u-x)}\right\}$$

$$= \frac{1}{\pi(a-x)}\log\left|\frac{a+\delta-x}{a-\delta-x}\right|.$$

Hence

$$\int_{\delta}^{\infty} \frac{f(a+x)-f(a-x)}{x}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\log\left|\frac{a+\delta-x}{a-\delta-x}\right|\frac{dx}{a-x}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} g(t+a)\log\left|\frac{\delta+t}{\delta-t}\right|\frac{dt}{t}. \tag{5.11.4}$$

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† M. Riesz (1), (2).

We can use (5.11.1) to give an alternative proof of the case of Theorem 101 where p is an odd integer.

Let $h(x)$, $k(x)$ be transforms of $L^{p'}$. Since the theorem has been proved for p', on making $b \to \infty$, $y' \to 0$ in (5.11.3) we obtain

$$\int_{-\infty}^{\infty} U_a(x,y)h(x)\,dx = \int_{-\infty}^{\infty} V_a(x,y)k(x)\,dx.$$

Hence

$$\left|\int_{-\infty}^{\infty} V_a(x,y)k(x)\,dx\right| \leqslant \left(\int_{-\infty}^{\infty} |U_a|^p\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |h(x)|^{p'}\,dx\right)^{1/p'}$$

$$\leqslant M_{p'}\left(\int_{-\infty}^{\infty} |f(t)|^p\,dt\right)^{1/p}\left(\int_{-\infty}^{\infty} |k(x)|^{p'}\,dx\right)^{1/p'}$$

by (5.10.4) and (5.10.3) for p'. Here $k(x)$ may be any function of $L^{p'}$. Take

$$k(x) = |V_a(x,y)|^{p-1}\operatorname{sgn} V_a(x,y).$$

Then
$$\int_{-\infty}^{\infty} |V_a|^p\,dx \leqslant M_{p'}\left(\int_{-\infty}^{\infty} |f(t)|^p\,dt\right)^{1/p}\left(\int_{-\infty}^{\infty} |V_a|^p\,dx\right)^{1/p'},$$

or
$$\int_{-\infty}^{\infty} |V_a|^p\,dx \leqslant M_{p'}^p \int_{-\infty}^{\infty} |f(t)|^p\,dt.$$

The theorem for p now follows as before.

It also follows that, if M_p is the least constant for which (5.10.3) holds, then $M_p \leqslant M_{p'}$. Hence, since p and p' are interchangeable, $M_p = M_{p'}$.

5.12. THEOREM 103. *Let $\Phi(z)$ be an analytic function, regular for $y > 0$, and let*

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^p\,dx < K \quad (p > 1) \tag{5.12.1}$$

for all values of y. Then $\Phi(x+iy)$ converges for almost all x, and also in the mean of order p, as $y \to 0$, to $f(x)-ig(x)$, where $f(x)$ and $g(x)$ are Hilbert transforms of the class L^p.

It is convenient to use the following lemma.

LEMMA. *Let $\lambda_n(x)$ be a sequence of functions such that*

$$\int_a^b |\lambda_n(x)|^p\,dx < K,$$

while $\lambda_n(x) \to 0$ almost everywhere. Then if $\mu(x)$ belongs to $L^{p'}$,

$$\int_a^b \lambda_n(x)\mu(x)\,dx \to 0.$$

Suppose first that the interval (a, b) is finite. By Egoroff's theorem† $\lambda_n(x) \to 0$ uniformly in a set E of measure $b-a-\delta$, and hence

$$\int\limits_E \lambda_n(x)\mu(x)\,dx \to 0.$$

Also

$$\left|\int\limits_{CE} \lambda_n(x)\mu(x)\,dx\right| \leqslant \left(\int\limits_{CE} |\lambda_n(x)|^p\,dx\right)^{1/p}\left(\int\limits_{CE} |\mu(x)|^{p'}\,dx\right)^{1/p'}$$

$$< K\left(\int\limits_{CE} |\mu(x)|^{p'}\,dx\right)^{1/p'},$$

which tends to 0 with δ, and is independent of n. Hence the result.

If $b = \infty$, we first take X so large that

$$\left|\int\limits_X^\infty \lambda_n(x)\mu(x)\,dx\right| \leqslant K\left(\int\limits_X^\infty |\mu(x)|^{p'}\,dx\right)^{1/p'} < \epsilon$$

and then argue as before with (a, X).

If $\Phi(z) \to f(x)-ig(x)$, it follows from Fatou's theorem that f and g belong to L^p. We prove (5.4.1) as before, and (5.4.2) now follows from (5.4.1) by the lemma, taking

$$\lambda_n(u) = \{\Phi(u+iv)-f(u)+ig(u)\}\frac{u-x+iy}{u-x+i(v-y)} \quad (n = v)$$

and

$$\mu(x) = \frac{1}{u-x+iy}.$$

Hence

$$\phi(z) = \tfrac{1}{2}(U+iV)-\tfrac{1}{2}i(P+iQ),$$

where U and V are (5.3.5), (5.3.6), and P, Q are defined similarly with g instead of f. Now make $y \to 0$. Denoting by f^* the conjugate of f, and by g^* that of g, we obtain

$$f-ig = \tfrac{1}{2}(f-if^*)-\tfrac{1}{2}i(g-ig^*).$$

Hence $f = -g^*$, $g = f^*$ almost everywhere.

That $\Phi(z)$ converges in mean to $f(x)-ig(x)$ with index p follows from the analysis of § 5.10.

That $\Phi(z)$ tends to a limit almost everywhere was deduced by Hille and Tamarkin (5) from the corresponding theorem for series (Zygmund, *Trigonometrical Series*, § 7.53). It could be proved directly as follows. If Φ has no zeros, the result follows on applying Theorem 93 to $\{\Phi(z)\}^{\frac{1}{2}p}$. Otherwise, let z_ν run through the zeros of Φ in $y > 0$, and let

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† Titchmarsh, *Theory of Functions*, p. 339.

$$B_n(z) = \prod_{\nu=1}^{n} \frac{z-z_\nu}{z-\bar{z}_\nu} \frac{\bar{z}_\nu+i}{z_\nu+i} \left|\frac{z_\nu+i}{\bar{z}_\nu+i}\right|$$

(assuming that no z_ν is i). For a fixed n, $|B_n| \geqslant 1-\epsilon$ for $y \leqslant \eta$, say, and all x. Let $\Phi(z) = G_n(z)B_n(z)$. Then

$$\int_{-\infty}^{\infty} |G_n(x+iy)|^p \, dx < \frac{K}{(1-\epsilon)^p}$$

for $y \leqslant \eta$. Since $G_n(x+iy')$, $G_n(x+iy)$, $(y < y')$ are related like the previous $\Phi(z)$, $f(x)-ig(x)$, it follows from the analysis of § 5.10 (especially (5.10.4)) that

$$\int_{-\infty}^{\infty} |G_n(x+iy)|^p \, dx < K'$$

for all y, K' depending on K and p only.

If Φ has an infinity of zeros, a little consideration of Carleman's formula (Titchmarsh, *Theory of Functions*, § 3.7) shows that

$$\sum \mathbf{I}(z_\nu)/(1+|z_\nu|^2)$$

is convergent, and hence that $B(z) = \lim B_n(z)$ and $G(z) = \lim G_n(z)$ exist and are analytic. It follows that $\Phi(z) = G(z)B(z)$, where $G(z)$ satisfies (5.12.1) with some K, and has no zeros, and $|B(z)| \leqslant 1$. Hence $G(z)$ tends to a limit almost everywhere, as before, and so does $B(z)$, by Theorem 94.

5.13. THEOREM 104. *Let $f(x)$ belong to L^p $(p > 1)$, and let $g(x)$ be its conjugate. Let $\lambda(x)$ belong to L^q, where $q > 1$, $pq \leqslant p+q$, and let*

$$h(x) = \int_{-\infty}^{\infty} \lambda(t)f(x-t) \, dt, \qquad k(x) = \int_{-\infty}^{\infty} \lambda(t)g(x-t) \, dt.$$

Then if $pq < p+q$, $h(x)$ and $k(x)$ are conjugates of the class L^P, where $P = pq/(p+q-pq)$. If $pq = p+q$, $h(x)$ and $k(x)$ are conjugates, in the sense that

$$k(x) = \frac{1}{\pi} \int_{\to 0}^{\to\infty} \frac{h(x+u)-h(x-u)}{u} \, du,$$

and reciprocally, for all values of x.

(i) Suppose first that $pq < p+q$. Then $h(x)$ and $k(x)$ belong to L^P, by Lemma β of § 4.2. We have to prove that they are conjugate.

Let $\Phi(z)$ be the analytic function of which $f(x)-ig(x)$ is the boundary value, and let

$$\Psi(z) = \int_{-\infty}^{\infty} \lambda(t)\Phi(z-t)\,dt \quad (y > 0). \tag{5.13.1}$$

It follows from the lemma of § 5.4 that $\Phi(z)$ is bounded in any strip $0 < y_1 \leqslant y \leqslant y_2$. Hence

$$\left|\int_{T}^{T'} \lambda(t)\Phi(z-t)\,dt\right| \leqslant \int_{T}^{T'} |\lambda(t)|\,.\,|\Phi(z-t)|^{p/q'}|\Phi(z-t)|^{1-p/q'}\,dt$$
$$\leqslant K(y_1,y_2)\left(\int_{T}^{T'} |\lambda(t)|^q\,dt\right)^{1/q}\left(\int_{T}^{T'} |\Phi(z-t)|^p\,dt\right)^{1/q'}.$$

Hence the integral (5.13.1) converges uniformly in $y_1 \leqslant y \leqslant y_2$. Hence $\Psi(z)$ is analytic for $y > 0$.

Also, by Lemma β of § 4.2,

$$\int_{-\infty}^{\infty} |\Psi(z)|^P\,dx \leqslant \left(\int_{-\infty}^{\infty} |\lambda(t)|^q\,dt\right)^{P/q}\left(\int_{-\infty}^{\infty} |\Phi(z-t)|^p\,dt\right)^{P/p},$$

which is bounded; and similarly

$$\int_{-\infty}^{\infty} |\Psi(z)-h(x)+ik(x)|^P\,dx$$
$$\leqslant \left(\int_{-\infty}^{\infty} |\lambda(t)|^q\,dt\right)^{P/q}\left(\int_{-\infty}^{\infty} |\Phi(z-t)-f(x-t)+ig(x-t)|^p\,dt\right)^{P/p},$$

which tends to 0 as $y \to 0$. Hence, by Theorem 103, $h(x)$ and $k(x)$ are conjugate.

(ii) If $pq = p+q$, it is known that $h(x)$ and $k(x)$ are continuous, and tend to 0 at infinity.†

In this case the integral defining $h(x)$ converges uniformly over any finite range. Hence

$$\frac{1}{\pi}\int_{\delta}^{\Delta} \frac{h(x+u)-h(x-u)}{u}\,du = \frac{1}{\pi}\int_{\delta}^{\Delta}\frac{du}{u}\int_{-\infty}^{\infty} \lambda(t)\{f(x+u-t)-f(x-u-t)\}\,dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \lambda(t)\,dt\int_{\delta}^{\Delta}\frac{f(x+u-t)-f(x-u-t)}{u}\,du$$
$$= \int_{-\infty}^{\infty} \lambda(t)\{g_\delta(x-t)-g_\Delta(x-t)\}\,dt,$$

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† See Titchmarsh, *Theory of Functions*, p. 398, exs. 20, 21.

where $$g_\delta(x) = \frac{1}{\pi}\int\limits_\delta^\infty \frac{f(x+t)-f(x-t)}{t}\,dt.$$

Now $$g_\delta(x-t) \to g(x-t), \qquad g_\Delta(x-t) \to 0$$
as $\delta \to 0$, $\Delta \to \infty$, for almost all t. Hence, by the lemma of § 5.12, it is sufficient to prove that

$$\int\limits_{-\infty}^{\infty} |g_\delta(x)|^p\,dx < K$$

for all δ. By (5.11.4)

$$g_\delta(x) = \frac{1}{\pi^2}\int\limits_{-\infty}^{\infty} g(t+x)\log\left|\frac{\delta+t}{\delta-t}\right|\frac{dt}{t}.$$

Hence

$$|g_\delta(x)|^p \leqslant \frac{1}{\pi^{2p}}\int\limits_{-\infty}^{\infty} |g(t+x)|^p\left|\log\left|\frac{\delta+t}{\delta-t}\right|\right|\frac{dt}{|t|}\left(\int\limits_{-\infty}^{\infty}\left|\log\left|\frac{\delta+t}{\delta-t}\right|\right|\frac{dt}{|t|}\right)^{p-1}$$

$$< K\int\limits_{-\infty}^{\infty} |g(t+x)|^p\left|\log\left|\frac{\delta+t}{\delta-t}\right|\right|\frac{dt}{|t|},$$

by putting $t = \delta u$ in the last factor. Integrating with respect to x and inverting, the result now follows.

5.14. The case $p = 1$. We have so far supposed that $f(x)$ belongs to L^p, where $p > 1$. The general Theorem 101 fails in the case $p = 1$, in which $f(x)$ belongs to L. We have seen (Theorem 100) that $g(x)$ still exists almost everywhere in this case. But $g(x)$ does not necessarily belong to L. Suppose for example that

$$f(t) = \frac{1}{t\log^2 t} \quad (t > 0), \qquad 0 \quad (t \leqslant 0).$$

Then for $x > 0$

$$g(-x) = \frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{f(t)}{t+x}\,dt > \frac{1}{\pi}\int\limits_0^x \frac{dt}{2xt\log^2 t} = \frac{1}{2\pi x\log x}.$$

Hence $g(x)$ does not belong to L. In fact it is possible to construct examples in which $g(x)$ does not belong to L over any interval, however small.

We have, however, the following theorem.[†]

[†] Corresponding to a theorem of Kolmogoroff on Fourier series; see Littlewood (1), Titchmarsh (13), Zygmund, *Trigonometrical Series*, § 7.24.

Theorem 105. *Let* $f(x)$ *belong to* $L(-\infty,\infty)$. *Then the formula*

$$g(x) = \frac{1}{\pi}\int\limits_{\to 0}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.14.1}$$

defines a finite $g(x)$ *for almost all values of* x, *and*

$$\int\limits_{-\infty}^{\infty} \frac{|g(x)|^p}{1+x^2}\,dx \tag{5.14.2}$$

is convergent if $0 < p < 1$.

We may suppose without loss of generality that $f(t) \geqslant 0$, and that $f(t)$ is not null. Let

$$\Phi_a(z) = \frac{1}{i\pi}\int\limits_{-a}^{a} \frac{f(t)}{t-z}\,dt = U_a+iV_a \quad (y > 0),$$

as before. Then

$$U_a(x,y) = \frac{y}{\pi}\int\limits_{-a}^{a} \frac{f(t)}{(t-x)^2+y^2}\,dt > 0$$

if a is large enough. Let $0 < p < 1$, and let

$$\{\Phi_a(z)\}^p = (U_a+iV_a)^p = e^{\frac{1}{2}p\log(U_a^2+V_a^2)+ip\arctan(V_a/U_a)},$$

where $-\frac{1}{2}\pi < \arctan(V_a/U_a) < \frac{1}{2}\pi$. For a fixed a, $\Phi_a(z) = O(1/|z|)$ as $|z| \to \infty$, and the calculus of residues gives

$$\int\limits_{-\infty}^{\infty} \frac{\{\Phi_a(z)\}^p}{z^2+1}\,dz = \pi\{\Phi_a(i)\}^p \quad (0 < y < 1).$$

Now
$$|\Phi_a(i)| = \left|\frac{1}{i\pi}\int\limits_{-a}^{a} \frac{f(t)}{t-i}\,dt\right| \leqslant \frac{1}{\pi}\int\limits_{-\infty}^{\infty} |f(t)|\,dt.$$

Hence
$$\left|\int\limits_{-\infty}^{\infty} \frac{(U_a+iV_a)^p}{z^2+1}\,dz\right| < K_p. \tag{5.14.3}$$

If $U > 0$, $|V| \geqslant 1$,

$$|(U+iV)^p-(iV)^p| = \left|p\int\limits_{iV}^{U+iV} z^{p-1}\,dz\right| \leqslant pU|V|^{p-1} \leqslant U,$$

while if $U > 0$, $|V| < 1$,

$$|(U+iV)^p-(iV)^p| \leqslant (U+1)^p+1 \leqslant U+2.$$

Hence

$$\left|\int_{-\infty}^{\infty}\frac{(U_a+iV_a)^p-(iV_a)^p}{z^2+1}\,dz\right| \leqslant \int_{-\infty}^{\infty}\frac{U_a+2}{x^2+1}\,dx$$

$$\leqslant \int_{-\infty}^{\infty}U_a\,dx+2\pi = \frac{y}{\pi}\int_{-\infty}^{\infty}dx\int_{-a}^{a}\frac{f(t)}{(t-x)^2+y^2}\,dt+2\pi$$

$$= \int_{-a}^{a}f(t)\,dt+2\pi \leqslant \int_{-\infty}^{\infty}f(t)\,dt+2\pi. \qquad (5.14.4)$$

From (5.14.3) and (5.14.4) it follows that

$$\left|\int_{-\infty}^{\infty}\frac{(iV_a)^p}{z^2+1}\,dz\right| < K_p.$$

Now

$$\mathbf{R}\left\{\frac{(iV_a)^p}{z^2+1}\right\} = |V_a|^p\frac{(x^2-y^2+1)\cos\frac{1}{2}p\pi\pm 2xy\sin\frac{1}{2}p\pi}{(x^2-y^2+1)^2+4x^2y^2} > K_p\frac{|V_a|^p}{x^2+1}$$

for sufficiently small y and all x. Hence

$$\int_{-\infty}^{\infty}\frac{|V_a|^p}{x^2+1}\,dx < K_p,$$

and the result follows as in the proof of Theorem 101.

5.15. Lipschitz conditions. THEOREM 106.[†] *Let $f(x)$ belong to L^p $(p > 1)$, and let it satisfy the Lipschitz condition*

$$|f(x+h)-f(x)| < K|h|^\alpha \quad (0 < \alpha < 1) \qquad (5.15.1)$$

uniformly in x, as $h \to 0$ (say for all x and $0 < h \leqslant 1$). Then Hilbert's reciprocal formulae (5.1.9), (5.1.10) *hold for all values of x; and $g(x)$ also belongs to L^p and satisfies a Lipschitz condition with the same α as $f(x)$.*

In this case the integral (5.1.9) plainly exists for all values of x.

We next observe that if $f(x)$ satisfies the given conditions then it is bounded—in fact it tends to 0 as $x \to \infty$. For since $f(x)$ is continuous, the points where $|f(x)| \geqslant \delta > 0$ form a set of intervals. The length of such an interval (x_1, x_2) tends to 0 as $x_1 \to \infty$, since

$$(x_2-x_1)\delta^p \leqslant \int_{x_1}^{x_2}|f(x)|^p\,dx \to 0.$$

[†] Titchmarsh (5). The result corresponds to Privaloff's theorem for Fourier series. See Zygmund, *Trigonometrical Series*, § 7.4.

Now since $|f(x_1)| = \delta$,

$$|f(x)| \leqslant |f(x_1)|+|f(x)-f(x_1)| \leqslant \delta+K|x-x_1|^\alpha \leqslant \delta+K|x_2-x_1|^\alpha,$$

which tends to 0, by choosing first δ and then x_1. Hence $f(x) \to 0$.

It follows that if (5.15.1) holds for small h it holds for all h, with a possible rechoice of the constant K.

Now as $y \to 0$,

$$V(x,y)+g(x) = \frac{1}{\pi}\int\limits_0^\infty \left(\frac{1}{t}-\frac{t}{t^2+y^2}\right)\{f(x+t)-f(x-t)\}\,dt$$

$$= \frac{y^2}{\pi}\int\limits_0^\infty \frac{f(x+t)-f(x-t)}{t(t^2+y^2)}\,dt = O\left(y^2\int\limits_0^\infty \frac{t^{\alpha-1}}{t^2+y^2}\,dt\right) = O(y^\alpha).$$

Also

$$\frac{\partial V}{\partial x} = -\frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{(t-x)^2-y^2}{\{(t-x)^2+y^2\}^2}f(t)\,dt$$

$$= -\frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{t^2-y^2}{(t^2+y^2)^2}f(t+x)\,dt = -\frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{t^2-y^2}{(t^2+y^2)^2}\{f(t+x)-f(x)\}\,dt$$

$$= O\left(\int\limits_{-\infty}^\infty \frac{|t^2-y^2|}{(t^2+y^2)^2}|t|^\alpha\,dt\right) = O(y^{\alpha-1}).$$

Hence, taking $h > 0$,

$$|g(x+h)-g(x)|$$

$$\leqslant |g(x+h)+V(x+h,h)|+|V(x+h,h)-V(x,h)|+|g(x)+V(x,h)|$$

$$= O(h^\alpha)+O\left(\left|\int\limits_x^{x+h}\frac{\partial}{\partial\xi}V(\xi,h)\,d\xi\right|\right)+O(h^\alpha) = O(h^\alpha),$$

so that $g(x)$ satisfies the required Lipschitz condition.

The reciprocal formula (5.1.10), already known to hold for almost all values of x, now holds for all values of x, since each side is continuous.

If $\alpha = 1$, we obtain similarly

$$\frac{\partial V}{\partial x} = O\left(\int\limits_{-1}^{1}\frac{|t^2-y^2|}{(t^2+y^2)^2}|t|\,dt\right)+O\left(\int\limits_1^\infty\frac{dt}{t^2}\right)$$

$$= O\left(\frac{1}{y}\int\limits_{-1/y}^{1/y}\frac{|u^2-1|}{u^2+1}|u|\,du\right)+O(1) = O\left(\frac{1}{y}\log\frac{1}{y}\right),$$

and it follows that

$$g(x+h)-g(x) = O(|h|\log 1/|h|).$$

5.16. The allied integral. We next return to the allied integral (5.1.7), which is formally equal to $g(x)$. We can now prove the following theorem.

THEOREM 107. *Let $f(x)$ belong to $L(-\infty,\infty)$. Then, for any positive α, the allied integral is summable (C,α) to $g(x)$ wherever*

$$g(x) = \frac{1}{\pi}\int_{\to 0}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.16.1}$$

exists, and
$$\int_0^h |f(x+t)-f(x-t)|\,dt = o(h); \tag{5.16.2}$$

and so almost everywhere.

It is plainly sufficient to suppose that $0<\alpha<1$.

We have to consider

$$\lim_{\lambda\to\infty}\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha du \int_{-\infty}^{\infty} f(x+t)\sin ut\,dt$$
$$= \lim_{\lambda\to\infty}\int_0^\infty \{f(x+t)-f(x-t)\}\,dt \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \sin ut\,du.$$

Now

$$\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \sin ut\,du = \lambda\int_0^1 (1-v)^\alpha \sin\lambda tv\,dv$$
$$= -\frac{1}{t}\Big[(1-v)^\alpha\cos\lambda tv\Big]_0^1+\frac{\alpha}{t}\int_0^1 (1-v)^{\alpha-1}\cos\lambda tv\,dv$$
$$= \frac{1}{t}+\frac{\alpha}{\lambda^\alpha t^{\alpha+1}}\int_0^{\lambda t}\frac{\cos(\lambda t-w)}{w^{1-\alpha}}\,dw.$$

Hence
$$\left|\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \sin ut\,du-\frac{1}{t}\right| < \frac{K(\alpha)}{\lambda^\alpha t^{\alpha+1}}.$$

It follows at once that, if $\delta>0$,

$$\lim_{\lambda\to\infty}\int_\delta^\infty \{f(x+t)-f(x-t)\}\,dt\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \sin ut\,du = \int_\delta^\infty \frac{f(x+t)-f(x-t)}{t}\,dt.$$

Also

$$\int_{1/\lambda}^{\delta}\{f(x+t)-f(x-t)\}\,dt\int_{0}^{\lambda}\left(1-\frac{u}{\lambda}\right)^{\alpha}\sin ut\,du$$

$$=\int_{1/\lambda}^{\delta}\frac{f(x+t)-f(x-t)}{t}\,dt+O\left\{\int_{1/\lambda}^{\delta}\frac{|f(x+t)-f(x-t)|}{t^{\alpha+1}\lambda^{\alpha}}\,dt\right\},$$

and, if (5.16.2) holds, the last term is

$$\left[\frac{1}{\lambda^{\alpha}t^{\alpha+1}}\int_{0}^{t}|f(x+u)-f(x-u)|\,du\right]_{1/\lambda}^{\delta}+$$

$$+\frac{(\alpha+1)}{\lambda^{\alpha}}\int_{1/\lambda}^{\delta}\frac{dt}{t^{\alpha+2}}\int_{0}^{t}|f(x+u)-f(x-u)|\,du$$

$$=o(1)+o\left(\frac{1}{\lambda^{\alpha}}\int_{1/\lambda}^{\delta}\frac{dt}{t^{\alpha+1}}\right)=o(1)$$

by choosing first δ then λ. Finally,

$$\left|\int_{0}^{\lambda}\left(1-\frac{u}{\lambda}\right)^{\alpha}\sin ut\,du\right|<K\lambda,$$

and

$$\left|\int_{0}^{1/\lambda}\{f(x+t)-f(x-t)\}\,dt\int_{0}^{\lambda}\left(1-\frac{u}{\lambda}\right)^{\alpha}\sin ut\,du\right|$$

$$<K\lambda\int_{0}^{1/\lambda}|f(x+t)-f(x-t)|\,dt=o(1).$$

This proves the theorem.

5.17. Application to Fourier Transforms. In this section we make an application of the theory of conjugate functions to the theory of Fourier transforms. There is one respect in which this theory is still incomplete. We have shown that if $f(x)$ belongs to L^{p} $(1<p<2)$ then $f(x)$ has a transform $F(x)$ belonging to $L^{p'}$, and $F(x)=\text{l.i.m.}\,F(x,a)$.

We have not yet been able to show that the reciprocal relation $f(x)=\text{l.i.m.}\,f(x,a)$, where

$$f(x,a)=\frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a}F(t)e^{-ixt}\,dt,$$

also holds. We can now supply this point.[†]

[†] Hille and Tamarkin (3).

THEOREM 108. *If $f(x)$ belongs to L^p $(1 < p < 2)$, then*

$$f(x) = \text{l.i.m.}\, f(x,a).$$

It follows from Parseval's formula (as in the proof of Theorem 58) that

$$f(x,a) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\frac{\sin(x-u)a}{x-u}\,du$$

$$= \frac{\sin xa}{\pi} P\int_{-\infty}^{\infty} \frac{f(u)\cos ua}{x-u}\,du - \frac{\cos xa}{\pi} P\int_{-\infty}^{\infty} \frac{f(u)\sin ua}{x-u}\,du$$

$$= \sin xa\,\phi_a(x) - \cos xa\,\psi_a(x)$$

say, the integrals being principal values at $u = x$. By Theorem 101

$$\int_{-\infty}^{\infty} |\phi_a(x)|^p\,dx < K_p \int_{-\infty}^{\infty} |f(u)\cos ua|^p\,du < K_p \int_{-\infty}^{\infty} |f(u)|^p\,du,$$

and similarly for $\psi_a(x)$. Hence

$$\int_{-\infty}^{\infty} |f(x,a)|^p\,dx \leqslant \int_{-\infty}^{\infty} (2^p|\phi_a(x)|^p + 2^p|\psi_a(x)|^p)\,dx < K_p \int_{-\infty}^{\infty} |f(u)|^p\,du.$$

This proves that $\displaystyle\int_{-\infty}^{\infty} |f(x)-f(x,a)|^p\,dx$

is bounded as $a \to \infty$. We have to prove that in fact it tends to zero.

We can construct a step-function $f^*(x)$, zero for $|x| > X$, and such that

$$\int_{-\infty}^{\infty} |f(x)-f^*(x)|^p\,dx < \epsilon.$$

Now

$$\left(\int_{-\infty}^{\infty} |f(x)-f(x,a)|^p\,dx\right)^{1/p} \leqslant \left(\int_{-\infty}^{\infty} |f(x)-f^*(x)|^p\,dx\right)^{1/p} +$$

$$+\left(\int_{-\infty}^{\infty} |f^*(x)-f^*(x,a)|^p\,dx\right)^{1/p} + \left(\int_{-\infty}^{\infty} |f^*(x,a)-f(x,a)|^p\,dx\right)^{1/p}$$

$$= J_1^{1/p} + J_2^{1/p} + J_3^{1/p},$$

say. By hypothesis, $|J_1| < \epsilon$; and by the above method

$$|J_3| < K_p|J_1| < K_p\,\epsilon.$$

Also $f^*(x,a) \to f^*(x)$ boundedly in any finite interval, say $(-\xi, \xi)$;

now if $\xi > 2X$,

$$\int_{\xi}^{\infty} |f^*(x)-f^*(x,a)|^p\,dx = \int_{\xi}^{\infty} \left| \sqrt{\left(\frac{2}{\pi}\right)} \int_{-X}^{X} f^*(t)\frac{\sin(x-t)a}{x-t}\,dt\right|^p dx$$

$$< K_p \int_{\xi}^{\infty} \left| \int_{-X}^{X} \frac{|f^*(t)|}{x-X}\,dt\right|^p dx < K_p \int_{\xi}^{\infty} \frac{dx}{(x-X)^p} \left(\int_{-X}^{X} |f^*(t)|\,dt\right)^p$$

We can choose ξ so large that this is less than ϵ, for all a; and, having fixed ξ,

$$\int_{-\xi}^{\xi} |f^*(x)-f^*(x,a)|^p\,dx \to 0$$

by bounded convergence. This completes the proof.

5.18. Further cases of Parseval's formula. We have already seen that (2.1.1) holds if f and G are L^p $(1 < p < 2)$. If f and g are the given functions, and belong to L^p, $L^{p'}$ respectively, we cannot state the result, because the existence of G is not known. We require an additional condition.

THEOREM 109. *If f is L^p $(1 < p < 2)$, and g is L^2 and $L^{p'}$, then*

$$\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)g(-x)\,dx.$$

Let G be the transform of g. Then

$$G(x)\ \ (|x| < \lambda),\quad 0\ \ (|x| > \lambda);\quad g(x,\lambda),$$

defined as in (3.1.2), are transforms of L^2; and the former is also L^p. They are therefore transforms of L^p, $L^{p'}$, and Theorem 75 gives

$$\int_{-\lambda}^{\lambda} F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)g(-x,\lambda)\,dx.$$

As in the previous section

$$\underset{\lambda\to\infty}{\text{l.i.m.}}(p')g(-x,\lambda) = g(-x),$$

and the result follows.

THEOREM 110. *If f is L^p $(1 < p < 2)$, g is $L^{p'}$, and the integral for G is uniformly convergent in any interval $0 < \delta \leqslant x \leqslant \lambda$, then*

$$\lim_{\delta\to 0,\,\lambda\to\infty} \left(\int_{-\lambda}^{-\delta} + \int_{\delta}^{\lambda} \right) F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)g(-x)\,dx.$$

We can now prove by uniform convergence that

$$G(x) \quad (\delta \leqslant |x| \leqslant \lambda), \qquad 0 \quad (|x| < \delta,\ |x| > \lambda)$$

is the transform of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} g(u)\frac{\sin\lambda(u-x)-\sin\delta(u-x)}{u-x}\,du.$$

The proof now goes as before, but to complete it we want

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} f(x)\,dx\int_{-\infty}^{\infty} g(u)\frac{\sin\delta(u-x)}{u-x}\,du = 0.$$

Now

$$\left|\int_{-\infty}^{\infty} g(u)\frac{\sin\delta(u-x)}{u-x}\,du\right|$$

$$\leqslant \delta\int_{x-1/\delta}^{x+1/\delta}|g(u)|\,du+\left(\int_{-\infty}^{x-1/\delta}+\int_{x+1/\delta}^{\infty}\right)\frac{|g(u)|}{|u-x|}\,du \to 0$$

as $\delta \to 0$, for any fixed x; and, as before, its mean p'th power is bounded. The result therefore follows from the lemma of § 5.12.

VI

UNIQUENESS AND MISCELLANEOUS THEOREMS

6.1. Uniqueness of trigonometrical integrals. THE classical uniqueness problem for trigonometrical series is to show that if

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=0$$

for all values of x in $(0,2\pi)$, or all values with some exceptions, then $a_n=0$, $b_n=0$ for all values of n.

The corresponding problem for integrals is to show that if

$$\int_0^{\infty}\{a(y)\cos xy+b(y)\sin xy\}\,dy=0 \tag{6.1.1}$$

in some sense or other for all values of x, possibly with some exceptions, then $a(y)=0$, $b(y)=0$ almost everywhere. A more general problem is to show that if a given function $f(x)$ is represented by a trigonometrical integral,

$$\int_0^{\infty}\{a(y)\cos xy+b(y)\sin xy\}\,dy=f(x), \tag{6.1.2}$$

in some sense, then the integral is necessarily of the Fourier form, i.e. in some sense

$$a(y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos xy\,dx,\qquad b(y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\sin xy\,dx.$$

Owing to the symmetry of the Fourier integral formula between a function and its transform, in the integral case this is not, formally, a new problem. It simply amounts to the question whether $a(x)$ and $b(x)$ are representable by Fourier integrals; and in some cases the answer follows from theorems which we have already proved.

Suppose, for example, that $a(x)$ and $b(x)$ belong to $L(0,\infty)$, and that

$$\int_0^{\infty}\{a(y)\cos xy+b(y)\sin xy\}\,dy=0$$

for almost all values of x. Adding and subtracting the formulae with x and $-x$, it follows that both

$$\int_0^{\infty}a(y)\cos xy\,dy=0,\qquad \int_0^{\infty}b(y)\sin xy\,dy=0$$

for almost all x. By Theorem 14 (for an even function)

$$\lim_{\lambda\to\infty}\frac{2}{\pi}\int\limits_0^\lambda\left(1-\frac{u}{\lambda}\right)\cos xu\,du\int\limits_0^\infty a(y)\cos uy\,dy = a(x)$$

for almost all x, and in this case the left-hand side is 0 for all x. Hence $a(x) = 0$ almost everywhere. Similarly $b(x) = 0$ almost everywhere.

Theorem 22 can be used to give the same result.

Suppose, again, that $a(y)$ belongs to $L^2(0,\infty)$, and that

$$\int\limits_0^{\to\infty} a(y)\cos xy\,dy = 0$$

for almost all values of x. Since the limit and mean limit of a sequence, if they both exist, are equal almost everywhere, the cosine transform of $a(x)$, in the sense of Theorem 48, is null, and hence $a(x)$ is the mean limit of a sequence of null functions, and is therefore null.

The uniqueness theory of Fourier series suggests a different type of theorem, in which the possible values of x for which (6.1.1) fails are much more restricted, but in which $a(x)$ and $b(x)$ do not necessarily belong to L-classes. The main difference between the theory for series and that for integrals is that the convergence of $\sum a_n \cos nx$, for example, in a set of positive measure, implies that $a_n \to 0$; but the convergence of

$$\int\limits_0^{\to\infty} a(y)\cos xy\,dy$$

does not imply that $a(x) \to 0$ as $x \to \infty$; for example, the integral is convergent if $a(x) = e^x \cos e^{2x}$.

6.2. The expression

$$\lim_{h\to 0}\frac{\Phi(x+h)+\Phi(x-h)-2\Phi(x)}{h^2} \tag{6.2.1}$$

is called the generalized second derivative of $\Phi(x)$. The uniqueness theory of Fourier series depends on the theorem of Schwarz (see Titchmarsh, *Theory of Functions*, § 13.84), that *if* $\Phi(x)$ *is continuous, and has at all points of an interval the generalized second derivative* 0, *then* $\Phi(x)$ *is a linear function in the interval.* Here we shall proceed at once to the general problem with $f(x)$, and use

THEOREM 111.[†] *Let* $\Phi(x)$ *be continuous in* (a, b), *and have at every point of this interval a finite generalized second derivative* $f(x)$, *which*

[†] De la Vallée-Poussin, *Cours d'analyse infinitésimale*, 1914 ed.

belongs to $L(a,b)$. *Then*

$$\Phi(x) = \int\limits_{a}^{x} du \int\limits_{a}^{u} f(v)\, dv + a_0 + a_1 x \quad (a \leqslant x \leqslant b), \tag{6.2.2}$$

where a_0 *and* a_1 *are constants.*

We first prove two lemmas.

LEMMA 1. *Let* $\Phi(x)$ *be continuous in* (a,b), *and let*

$$\overline{\lim_{h\to 0}} \frac{\Phi(x+h)+\Phi(x-h)-2\Phi(x)}{h^2} \tag{6.2.3}$$

be $\geqslant 0$ *for every* x *in* (a,b). *Then no part of any arc of the curve* $y = \Phi(x)$ *can be above its chord.*

Suppose that some points of an arc (x_1, x_2) lie above the chord, PQ say. Let

$$\Phi_\epsilon(x) = \Phi(x) + \tfrac{1}{2}\epsilon(x-x_1)(x-x_2) \quad (\epsilon > 0).$$

Then, if ϵ is small enough, some points of the corresponding arc of $y = \Phi_\epsilon(x)$ will lie above the chord. Let M be such a point of this curve, whose distance from PQ is not less than that of any other such point. Let x be the abscissa of M. Then, if λ is the tangent of the angle which PQ makes with the x-axis,

$$\frac{\Phi_\epsilon(x+h)-\Phi_\epsilon(x)}{h} \leqslant \lambda, \qquad \frac{\Phi_\epsilon(x)-\Phi_\epsilon(x-h)}{h} \geqslant \lambda.$$

Hence
$$\Phi_\epsilon(x+h)+\Phi_\epsilon(x-h)-2\Phi_\epsilon(x) \leqslant 0,$$
i.e.
$$\Phi(x+h)+\Phi(x-h)-2\Phi(x) \leqslant -\epsilon,$$
for all small h. This contradicts the hypothesis, and the result follows.

LEMMA 2. *Let* $\Phi(x)$ *be continuous in* (a,b), *and let* (6.2.3) *be* $\geqslant 0$ *for almost all* x *in* (a,b) *and be nowhere* $-\infty$. *Then no part of any arc of the curve* $y = \Phi(x)$ *can be above its chord.*

If (6.2.3) is nowhere < 0, the result follows from the previous lemma. Otherwise, let E be the set, of measure 0, where (6.2.3) < 0. Let† $\chi(x)$ be a non-decreasing absolutely continuous function such that $\chi'(x) = +\infty$ in E, and $\chi(b)-\chi(a) < \epsilon$. Let

$$\chi_1(x) = \int\limits_{a}^{x} \chi(u)\, du.$$

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† See e.g. Titchmarsh, *Theory of Functions*, § 11.83.

If x is a point of E, and M any positive number, however large,

$$\frac{\chi(x+u)-\chi(x-u)}{2u} \geqslant M \quad (|u| \leqslant \delta).$$

Hence, if $h \leqslant \delta$,

$$\frac{\chi_1(x+h)+\chi_1(x-h)-2\chi_1(x)}{h^2} = \frac{1}{h^2}\int\limits_0^h \{\chi(x+u)-\chi(x-u)\}\,du$$

$$\geqslant \frac{1}{h^2}\int\limits_0^h 2uM\,du = M,$$

and so the left-hand side tends to infinity as $h \to 0$.

Let $$\Omega(x) = \Phi(x)+\chi_1(x).$$

Then $$\overline{\lim_{h\to 0}}\frac{\Omega(x+h)+\Omega(x-h)-2\Omega(x)}{h^2} \geqslant 0$$

for every x in (a,b). Hence no part of any arc of $y = \Omega(x)$ can be above its chord. Since this is true for arbitrarily small ϵ, the same result follows for $y = \Phi(x)$.

Proof of Theorem 111.

Let $$p(x) = \min\{f(x), n\}, \qquad q(x) = \max\{f(x), -n\}.$$

Then $p(x) \leqslant f(x) \leqslant q(x)$, and, since f is integrable, so are p and q. Let

$$p_1(x) = \int\limits_a^x p(u)\,du, \qquad p_2(x) = \int\limits_a^x p_1(u)\,du,$$

and similarly for q.

Then $q_2(x)-\Phi(x)$ has almost everywhere the generalized second derivative $q(x)-f(x) \geqslant 0$, and

$$\frac{q_2(x+h)+q_2(x-h)-2q_2(x)}{h^2} = \frac{1}{h^2}\int\limits_0^h du \int\limits_{x-u}^{x+u} q(v)\,dv$$

$$\geqslant \frac{1}{h^2}\int\limits_0^h du \int\limits_{x-u}^{x+u} (-n)\,dv = -n.$$

Hence the generalized second derivative of $q_2(x)-\Phi(x)$ is nowhere $-\infty$. Hence no arc of $y = q_2(x)-\Phi(x)$ is above its chord.

The chord through the end-points a and b is

$$y = \frac{x-a}{b-a}\{q_2(b)-\Phi(b)+\Phi(a)\}-\Phi(a),$$

and hence

$$q_2(x)-\Phi(x) \leqslant \frac{x-a}{b-a}\{q_2(b)-\Phi(b)+\Phi(a)\}-\Phi(a) \quad (a \leqslant x \leqslant b).$$

Similarly, no arc of $y = p_2(x)-\Phi(x)$ is below its chord, and it follows that

$$p_2(x)-\Phi(x) \geqslant \frac{x-a}{b-a}\{p_2(b)-\Phi(b)+\Phi(a)\}-\Phi(a) \quad (a \leqslant x \leqslant b).$$

Making $n \to \infty$, $p_2(x)$ and $q_2(x)$ both tend to the limit

$$f_2(x) = \int_a^x du \int_a^u f(v)\, dv.$$

Hence

$$f_2(x)-\Phi(x) = \frac{x-a}{b-a}\{f_2(b)-\Phi(b)+\Phi(a)\}-\Phi(a) \quad (a \leqslant x \leqslant b),$$

the desired result.

6.3. THEOREM 112. *Let $a(y)$ and $b(y)$ be integrable over any finite interval, and zero in an interval containing the origin. Let*

$$\int_0^{\to\infty} \{a(y)\cos xy + b(y)\sin xy\}\, dy = f(x)$$

for all x in a certain interval. Then

$$\Phi(x) = -\int_0^{\to\infty} \{a(y)\cos xy + b(y)\sin xy\} \frac{dy}{y^2}$$

exists for every x of the interval, and has the generalized second derivative $f(x)$.

The convergence of the integral for $\Phi(x)$ follows from the second mean-value theorem. Now

$$\frac{\cos}{\sin}(x+h)y + \frac{\cos}{\sin}(x-h)y - 2\frac{\cos}{\sin}xy = -4\sin^2\tfrac{1}{2}hy \frac{\cos}{\sin}xy.$$

Hence

$$\frac{\Phi(x+h)+\Phi(x-h)-2\Phi(x)}{h^2} = \int_0^{\to\infty} \{a(y)\cos xy + b(y)\sin xy\} \frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\, dy, \tag{6.3.1}$$

and it is sufficient to prove that this integral converges uniformly with respect to h for $h \geqslant 0$.

Let $$\int_Y^{\to\infty} \{a(y)\cos xy + b(y)\sin xy\}\, dy = r(Y),$$

so that $|r(Y)| \leqslant \epsilon$ for $Y \geqslant Y_0(\epsilon)$. Then

$$\left|\int_{Y_0}^{\infty} \{a(y)\cos xy+b(y)\sin xy\} \frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\, dy\right|$$

$$= \left|\left[-r(y)\frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\right]_{Y_0}^{\infty} + \int_{Y_0}^{\infty} r(y)\frac{d}{dy}\left(\frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\right) dy\right|$$

$$\leqslant \epsilon\frac{4\sin^2\frac{1}{2}hY_0}{h^2Y_0^2}+\epsilon\int_{Y_0}^{\infty}\left|\frac{d}{dy}\left(\frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\right)\right| dy$$

$$< \epsilon+\epsilon\int_{0}^{\infty}\left|\frac{d}{du}\left(\frac{4\sin^2\frac{1}{2}u}{u^2}\right)\right| du = A\epsilon$$

for all $h > 0$; hence the result.

6.4. We shall now prove the uniqueness theorem on the assumption that $a(y)/(1+y^2)$ and $b(y)/(1+y^2)$ belong to $L(0,\infty)$. Later it will be shown that this condition is superfluous.

THEOREM 113.[†] *Let $a(y)/(1+y^2)$ and $b(y)/(1+y^2)$ belong to $L(0,\infty)$, and let*

$$\int_{0}^{\to\infty} \{a(y)\cos xy+b(y)\sin xy\}\, dy = f(x) \tag{6.4.1}$$

for all values of x, where $f(x)$ is everywhere finite and integrable over any finite interval. Then for almost all positive values of y

$$a(y) = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)f(x)\cos xy\, dx, \tag{6.4.2}$$

$$b(y) = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)f(x)\sin xy\, dx. \tag{6.4.3}$$

In particular, if $f(x) = 0$, then $a(y) = 0$, $b(y) = 0$ almost everywhere.

The condition (6.4.1) may be broken for a finite set of values of x, provided that $a(y) \to 0$, $b(y) \to 0$, as $y \to \infty$.

By replacing x by $-x$ in (6.4.1) and adding or subtracting, we obtain similar formulae with the cosine or sine integral only. We may therefore consider them separately.

[†] Pollard (3), Jacob (2). The unrestricted result follows from Offord (7); the proof given here is by Offord and the author.

Suppose first then that $b(y) = 0$. Suppose also that $a(y) = 0$ for $0 < y < \delta$, and let

$$\Phi(x) = -\int\limits_0^\infty \frac{a(y)}{y^2}\cos xy\,dy. \tag{6.4.4}$$

By Theorem 112, $\Phi(x)$ has the generalized second derivative $f(x)$, and here $f(x)$ is integrable. Hence, by Theorem 111,

$$\Phi(x) = \int\limits_0^x du \int\limits_0^u f(v)\,dv + p + qx,$$

where p and q are constants; and $q = 0$ in this case, since Φ and f are even. Writing

$$f_1(u) = \int\limits_0^u f(v)\,dv,$$

we have
$$\Phi(x) = \int\limits_0^x f_1(u)\,du + p,$$

and

$$\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)f(x)\cos xy\,dx = \int\limits_0^\lambda f_1(x)\left\{\frac{\cos xy}{\lambda} + \left(1-\frac{x}{\lambda}\right)y\sin xy\right\}dx$$

$$= \left[\Phi(x)\left\{\frac{\cos xy}{\lambda} + \left(1-\frac{x}{\lambda}\right)y\sin xy\right\}\right]_0^\lambda +$$

$$+ \int\limits_0^\lambda \Phi(x)\left\{\frac{2y\sin xy}{\lambda} - y^2\left(1-\frac{x}{\lambda}\right)\cos xy\right\}dx$$

$$= \frac{\Phi(\lambda)\cos\lambda y}{\lambda} - \frac{p}{\lambda} + \frac{2y}{\lambda}\int\limits_0^\lambda \Phi(x)\sin xy\,dx - y^2\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)\Phi(x)\cos xy\,dx. \tag{6.4.5}$$

Since $a(y)/y^2$ is L, (6.4.4) and Theorem 14 give

$$\lim_{\lambda\to\infty}\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)\Phi(x)\cos xy\,dx = -\tfrac{1}{2}\pi\frac{a(y)}{y^2} \tag{6.4.6}$$

almost everywhere. Also $\Phi(x) \to 0$ as $x \to \infty$, by Theorem 1, so that the remaining terms in (6.4.5) tend to 0 as $\lambda \to \infty$. Hence

$$\lim_{\lambda\to\infty}\int\limits_0^\lambda \left(1-\frac{x}{\lambda}\right)f(x)\cos xy\,dx = \tfrac{1}{2}\pi a(y), \tag{6.4.7}$$

the required result with the conditions stated.

To remove the restriction that $a(y) = 0$ over $(0,\delta)$, let

$$a_\delta(x) = a(x) \quad (x \geqslant \delta), \qquad 0 \quad (x < \delta),$$

and let

$$\int_0^\delta a(y)\cos xy\,dy = \chi(x).$$

Then the result already obtained shows that

$$\lim_{\lambda\to\infty}\int_0^\lambda \left(1-\frac{x}{\lambda}\right)\{f(x)-\chi(x)\}\cos xy\,dx = \tfrac{1}{2}\pi a_\delta(y)$$

almost everywhere. Also, by Theorem 14,

$$\lim_{\lambda\to\infty}\int_0^\lambda \left(1-\frac{x}{\lambda}\right)\chi(x)\cos xy\,dx = 0$$

for almost all y in (δ,∞). Hence

$$\lim_{\lambda\to\infty}\int_0^\lambda \left(1-\frac{x}{\lambda}\right)f(x)\cos xy\,dx = \tfrac{1}{2}\pi a(y)$$

for almost all y in (δ,∞), and so, since δ is arbitrary, for almost all y in $(0,\infty)$. This is the required theorem for the cosine integral.

Next let $a(y) = 0$, and suppose that $b(y) = 0$ in $(0,\delta)$. Let

$$\Psi(x) = -\int_0^\infty \frac{b(y)}{y^2}\sin xy\,dy. \tag{6.4.8}$$

Arguing as before, we obtain

$$\Psi(x) = \int_0^x du\int_0^u f(v)\,dv + qx,$$

and

$$\int_0^\lambda \left(1-\frac{x}{\lambda}\right)f(x)\sin xy\,dx = \int_0^\lambda \{f_1(x)+q\}\left\{\frac{\sin xy}{\lambda}-\left(1-\frac{x}{\lambda}\right)y\cos xy\right\}dx$$

$$= \frac{\Psi(\lambda)\sin\lambda y}{\lambda}-\frac{2y}{\lambda}\int_0^\lambda \Psi(x)\cos xy\,dx - y^2\int_0^\lambda \left(1-\frac{x}{\lambda}\right)\Psi(x)\sin xy\,dx. \tag{6.4.9}$$

The proof now concludes as in the cosine case.

If there are exceptional points where (6.4.1) does not hold, the argument merely shows that

$$y = \Phi(x) - \int_0^x du\int_0^u f(v)\,dv \tag{6.4.10}$$

is a linear function in the intervals between these points. But now, if

$$|a(y)| \leqslant \epsilon,\ |b(y)| \leqslant \epsilon \quad \text{for} \quad y \geqslant \Delta,$$

$$\left|\int\limits_0^\infty \{a(y)\cos xy + b(y)\sin xy\}\frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\,dy\right|$$

$$\leqslant \int\limits_0^\Delta \{|a(y)|+|b(y)|\}\,dy + 2\epsilon\int\limits_\Delta^\infty \frac{4\sin^2\frac{1}{2}hy}{h^2y^2}\,dy$$

$$< K(\Delta)+A\epsilon/h.$$

Multiplying (6.3.1) by h, and choosing first ϵ and then h, it follows that

$$\lim_{h\to 0}\left\{\frac{\Phi(x+h)-\Phi(x)}{h}-\frac{\Phi(x)-\Phi(x-h)}{h}\right\} = 0.$$

Taking x to be one of the exceptional points, it follows that the slopes of the straight lines which make up the graph of (6.4.10) are the same on each side of the point. Hence (6.4.10) is a single linear function, and the result then follows as before.

6.5. To remove the restriction on $a(y)$ and $b(y)$ we require some more preliminary theorems.

THEOREM 114. *If*

$$\int\limits_0^{\to\infty} f(t)\cos yt\,dt$$

converges uniformly in any finite interval, to $\sqrt{(\frac{1}{2}\pi)}F_c(y)$ *say, then*

$$\lim_{\lambda\to\infty}\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\lambda \left(1-\frac{y}{\lambda}\right)F_c(y)\cos xy\,dy = f(x)$$

for almost all x.

This is merely a variant of Theorem 20; the proof is substantially the same, depending on the particular case of the data, that

$$\int\limits_0^{\to\infty} f(t)\,dt$$

exists.

We require a similar theorem for the sine integral, but in this case the argument is more complicated.

THEOREM 115.[†] *Let* x_n *be a sequence of numbers tending to infinity, such that*

$$x_n \geqslant k^{n-1}x_{n-1} \quad (n = 2, 3,\ldots),$$

[†] Cantor (1).

where $k > 1$. Then, given any interval (α, β), there is a number Ω in (α, β), and a sequence of integers $y_1, y_2, \ldots$, such that

$$x_n \Omega - (2y_n + 1) \to 0. \tag{6.5.1}$$

Suppose that $0 < \alpha < \beta$, and divide (α, β) into three equal parts (α, γ), (γ, δ), (δ, β). Let x_ν be the first of the x_n which is greater than both

$$\frac{3}{(k-1)(\beta-\alpha)} \quad \text{and} \quad \frac{6}{\beta-\alpha}.$$

Choose y_ν so that $(2y_\nu + 1)/x_\nu$ falls in (γ, δ). This is possible, since $x_\nu > 6/(\beta-\alpha)$. Then determine $y_{\nu+1}, y_{\nu+2}, \ldots$ so that

$$\left|(2y_{n+1}+1) - (2y_n+1)\frac{x_{n+1}}{x_n}\right| \leqslant 1 \quad (n = \nu, \nu+1, \ldots).$$

If in any case there is more than one such y_{n+1}, take the least; the process is then unique; $y_1, \ldots, y_{\nu-1}$ can have any values.

The numbers x_n and y_n now determine a sequence of fractions $(2y_n + 1)/x_n$, which tend to a limit Ω; for, by the above inequality,

$$\left|\frac{2y_{n+m}+1}{x_{n+m}} - \frac{2y_n+1}{x_n}\right| \leqslant \frac{1}{x_{n+1}} + \ldots + \frac{1}{x_{n+m}} \to 0.$$

Also

$$\left|\Omega - \frac{2y_n+1}{x_n}\right| \leqslant \frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} + \ldots \leqslant \frac{1}{x_n}\left(\frac{1}{k^n} + \frac{1}{k^{2n}} + \ldots\right) = \frac{1}{x_n(k^n-1)},$$

so that (6.5.1) follows. Finally, Ω is in (α, β), since

$$\left|\Omega - \frac{2y_\nu+1}{x_\nu}\right| \leqslant \frac{1}{x_\nu(k^\nu-1)} \leqslant \frac{1}{x_\nu(k-1)} < \tfrac{1}{3}(\beta-\alpha).$$

Theorem 116. *If*

$$\int_0^{\to\infty} \phi(u) \sin xu \, du$$

is convergent for all values of x, then as $n \to \infty$

$$r_n = \max_{0 \leqslant \xi \leqslant 1} \int_n^{n+\xi} \phi(u)\, du \to 0.$$

If the theorem is false, there is a positive ϵ and a sequence n_ν such that $|r_{n_\nu}| \geqslant \epsilon$; and from this sequence we can select a sub-sequence n_μ satisfying $n_\mu \geqslant 2^{\mu-1} n_{\mu-1}$. Hence there is an x in $(0, \frac{1}{8}\pi)$, and integers $y_1, y_2, \ldots$, such that

$$2xn_\mu/\pi - (2y_\mu + 1) \to 0.$$

Hence, if μ is large enough,

$$|xn_\mu - (y_\mu + \tfrac{1}{2})\pi| \leqslant \tfrac{1}{8}\pi.$$

Hence, for $n_\mu \leqslant u \leqslant n_\mu+\xi$, where $\xi \leqslant 1$, $xu-y_\mu \pi$ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$; hence $\sin xu$ is monotonic in at most two intervals, and $|\sin xu| \geqslant 1/\sqrt{2}$. If e.g. $\sin xu$ is steadily increasing,

$$\int\limits_{n_\mu}^{n_\mu+\xi} \phi(u)\,du = \int\limits_{n_\mu}^{n_\mu+\xi} \frac{\phi(u)\sin xu}{\sin xu}\,du = \frac{1}{\sin xn_\mu}\int\limits_{n_\mu}^{n_\mu+\xi'} \phi(u)\sin xu\,du$$

by the second mean-value theorem, and the right-hand side tends to 0. Similarly in other cases.

Theorem 117. *If*

$$\int\limits_0^{\to\infty} f(t)\sin yt\,dt \tag{6.5.2}$$

converges uniformly in every finite interval, to $\sqrt{(\frac{1}{2}\pi)}F_s(y)$ *say, then*

$$\lim_{\lambda\to\infty}\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^{\lambda}\left(1-\frac{y}{\lambda}\right)F_s(y)\sin xy\,dy = f(x) \tag{6.5.3}$$

for almost all x.

We can insert (6.5.2) in (6.5.3) and invert, by uniform convergence. We obtain

$$\frac{1}{\pi}\int\limits_0^{\to\infty} f(t)\left\{\frac{1-\cos\lambda(x-t)}{\lambda(x-t)^2}-\frac{1-\cos\lambda(x+t)}{\lambda(x+t)^2}\right\}dt. \tag{6.5.4}$$

Let $T > x$, and consider

$$I = \int\limits_T^{\to\infty} \frac{f(t)\cos\lambda t}{\lambda(x-t)^2}\,dt = \left(\int\limits_T^{\frac{1}{2}N\pi/\lambda} + \sum_{n=N}^{\infty}\int\limits_{\frac{1}{2}n\pi/\lambda}^{\frac{1}{2}(n+1)\pi/\lambda}\right)\frac{f(t)\cos\lambda t}{\lambda(x-t)^2}\,dt,$$

where N is the integer next above $2\lambda T/\pi$. By the second mean-value theorem

$$\int\limits_{\frac{1}{2}n\pi/\lambda}^{\frac{1}{2}(n+1)\pi/\lambda} \frac{f(t)\cos\lambda t}{\lambda(x-t)^2}\,dt = O\left\{\frac{1}{\lambda(n/\lambda)^2}\int\limits_\xi^\eta f(t)\,dt\right\},$$

where $\frac{1}{2}n\pi/\lambda \leqslant \xi < \eta \leqslant \frac{1}{2}(n+1)\pi/\lambda$. By Theorem 116 the last integral tends to 0 as $n\to\infty$, uniformly for $\lambda > 2/\pi$. Hence

$$I = o\left(\frac{\lambda}{N^2}+\sum_{n=N}^{\infty}\frac{\lambda}{n^2}\right) = o\left(\frac{\lambda}{N}\right) = o\left(\frac{1}{T}\right)$$

uniformly with respect to λ. Similar arguments apply to the rest of (6.5.4) with $t \geqslant T$. Also, for a fixed T, the part with $t < T$ tends to $f(x)$ almost everywhere. The result therefore follows.

6.6. THEOREM 118. *The results of Theorem* 113 *hold if $a(y)$ and $b(y)$ are integrable over any finite interval, and* (6.4.1) *holds.*

We again define $\Phi(x)$ by (6.4.4), but now the integral is not necessarily absolutely convergent, and $\Phi(x)$ does not necessarily tend to 0 at infinity. However, since

$$a_1(y) = \int_0^y a(u)\,du \to \text{limit}$$

as $y \to \infty$ (by putting $x = 0$ in (6.4.1)), and

$$\int_0^Y \frac{a(y)}{y^2}\cos xy\,dy = \frac{a_1(Y)\cos xY}{Y^2} + \int_0^Y a_1(y)\left\{\frac{x\sin xy}{y^2} + \frac{2\cos xy}{y^3}\right\}dy,$$

the integral (6.4.4) converges uniformly over any finite interval. We therefore deduce (6.4.6) from Theorem 114. Also

$$\Phi(x) = -\int_0^\infty a_1(y)\left\{\frac{x\sin xy}{y^2} + \frac{2\cos xy}{y^3}\right\}dy = o(x)$$

as $x \to \infty$, by Theorem 1; and, for a fixed y,

$$\int_0^\lambda \Phi(x)\sin xy\,dx = \int_0^\lambda \left\{o(1) - \int_{y+1}^\infty a_1(u)\frac{x\sin xu}{u^2}\,du\right\}\sin xy\,dx$$

$$= o(\lambda) - \tfrac{1}{2}\int_{y+1}^\infty \frac{a_1(u)}{u^2}\left\{\frac{\lambda\sin\lambda(u-y)}{u-y} - \frac{1-\cos\lambda(u-y)}{u-y} - \right.$$

$$\left. - \frac{\lambda\sin\lambda(u+y)}{u+y} + \frac{1-\cos\lambda(u+y)}{u+y}\right\}du = o(\lambda)$$

as $\lambda \to \infty$. Hence (6.4.7) again follows from (6.4.5).

In the sine case we obtain (6.4.8) as before, but now we get no result by putting $x = 0$ in (6.4.1), and we have to use Theorems 115–17.

We have

$$\int_{y_1}^{y_2} \frac{b(y)}{y^2}\sin xy\,dy = \left(\int_{y_1}^{\frac{1}{2}m\pi/x} + \sum_{\nu=m}^{n}\int_{\frac{1}{2}\nu\pi/x}^{\frac{1}{2}(\nu+1)\pi/x} + \int_{\frac{1}{2}(n+1)\pi/x}^{y_2}\right)\frac{b(y)}{y^2}\sin xy\,dy.$$

The second mean-value theorem gives

$$\int_{\frac{1}{2}\nu\pi/x}^{\frac{1}{2}(\nu+1)\pi/x} \frac{b(y)}{y^2}\sin xy\,dy = O\left(\frac{x^2}{\nu^2}\int_\xi^\eta b(y)\,dy\right),$$

where $\frac{1}{2}\nu\pi/x \leqslant \xi < \eta \leqslant \frac{1}{2}(\nu+1)\pi/x$; and the last integral is $o(1)$ as $\nu \to \infty, x \to \infty$, and is $o(1/x)$ as $\nu \to \infty$, $x \to 0$ (by considering $O(1/x)$ terms of type r_n). Hence as x tends to 0 or ∞

$$\int\limits_{y_1}^{y_2} \frac{b(y)}{y^2} \sin xy\, dy = o\left\{x^2\left(1+\frac{1}{x}\right)\frac{1}{m}\right\} = o\left(\frac{x+1}{y_1}\right).$$

Hence (6.4.8) converges uniformly over any finite interval, and

$$\Psi(x) = o(x)$$

as $x \to \infty$. Also

$$\int\limits_0^\lambda \Psi(x) \cos xy\, dx$$

$$= o(\lambda) - \frac{1}{2} \int\limits_{y+1}^{\to\infty} \frac{b(u)}{u^2}\left\{\frac{1-\cos\lambda(u+y)}{u+y} + \frac{1-\cos\lambda(u-y)}{u-y}\right\} du = o(\lambda)$$

as $\lambda \to \infty$, by an argument similar to that just used for $\Phi(x)$. The result now follows as in the previous case.

We can also state the above result as a direct theorem on Fourier's integral formula.

THEOREM 119. *Let $f(x)$ be integrable over any finite interval, and let*

$$\lim_{\mu\to\infty} \int\limits_{-\mu}^{\mu} f(t)\cos xt\, dt, \qquad \lim_{\mu\to\infty} \int\limits_{-\mu}^{\mu} f(t)\sin xt\, dt$$

have finite values for every x, and let these values be integrable over any finite interval. Then

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{\pi} \int\limits_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right)\left\{\lim_{\mu\to\infty} \int\limits_{-\mu}^{\mu} f(t)\cos u(x-t)\, dt\right\} du.$$

6.7. Integrals in the complex form.

The result in this form is

THEOREM 120. *Let $F(y)$ be integrable over any finite interval, and let*

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\lambda}^{\lambda} F(y)e^{-ixy}\, dy = f(x) \tag{6.7.1}$$

for all values of x, where $f(x)$ is everywhere finite and integrable over any finite interval. Then for almost all y

$$F(y) = \lim_{\lambda\to\infty} \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\lambda}^{\lambda} \left(1-\frac{|x|}{\lambda}\right) f(x)e^{ixy}\, dx. \tag{6.7.2}$$

In particular, if $f(x) = 0$, then $F(y) = 0$ almost everywhere.

Since

$$\int_{-\lambda}^{\lambda} F(y)e^{-ixv}\,dy$$

$$= 2\int_{0}^{\lambda} [\{F(y)+F(-y)\}\cos xy - i\{F(y)-F(-y)\}\sin xy]\,dy,$$

the theorem is equivalent to Theorem 118.

There is a specially simple argument† for the case $f(x) = 0$. Let

$$F_1(x) = \int_{0}^{x} F(u)\,du.$$

Then

$$\int_{-\lambda}^{\lambda} F(y)e^{-ixv}\,dy = F_1(\lambda)e^{-ix\lambda} - F_1(-\lambda)e^{ix\lambda} + ix\int_{-\lambda}^{\lambda} F_1(y)e^{-ixv}\,dy.$$

Replacing x by $-x$, and adding,

$$\int_{-\lambda}^{\lambda} F(y)\cos xy\,dy = \{F_1(\lambda)-F_1(-\lambda)\}\cos x\lambda + x\int_{-\lambda}^{\lambda} F_1(y)\sin xy\,dy.$$

Now $$F_1(\lambda)-F_1(-\lambda) = \int_{-\lambda}^{\lambda} F(y)\,dy \to 0$$

as $\lambda \to \infty$, as a particular case of the data. Hence

$$\lim_{\lambda\to\infty} x\int_{-\lambda}^{\lambda} F_1(y)\sin xy\,dy = 0,$$

and so $$\lim_{\lambda\to\infty} \int_{0}^{\lambda} \{F_1(y)-F_1(-y)\}\sin xy\,dy = 0$$

for $x \neq 0$, by the above argument; and for $x = 0$ because the integrand is 0.

We deduce that $$F_1(y)-F_1(-y) = 0$$

from Theorem 118; but now $b(y)$ is bounded, and all that we want about $\Psi(x)$ is obvious from its definition. Hence

$$F(y)+F(-y) \equiv 0.$$

The argument applies equally well with $F(y)e^{i\xi v}$, with any ξ, instead of $F(y)$. Hence

$$F(y)e^{i\xi v}+F(-y)e^{-i\xi v} \equiv 0.$$

for every y and ξ. Hence $F(y) \equiv 0$.

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† Offord (6).

Offord (7) has recently proved the remarkable theorem that *if* $F(y)$ *is integrable over every finite interval, and* (6.7.1) *is summable* $(C,1)$ *to* 0 *for all* x, *i.e.*

$$\lim_{\lambda\to 0}\int_{-\lambda}^{\lambda}\left(1-\frac{|y|}{\lambda}\right)F(y)e^{-ixv}\,dy=0$$

for all x, *then* $F(y)=0$ *almost everywhere.*

This theorem is a 'best possible' both in the sense that one exceptional point is sufficient to render the conclusion false, and in the sense that it is not possible to replace $(C,1)$ by $(C,1+\delta)$ for any positive δ. Thus

$$\int_{-\infty}^{\infty} e^{ixu}\,du=0\quad (C,1)\quad (x\neq 0),$$

and

$$\int_{-\infty}^{\infty} ue^{ixu}\,du=0\quad (C,1+\delta)$$

for all x.

6.8. Parseval's formula.

The above results enable us to prove still another theorem on Parseval's formula.

Suppose that f and g are given functions, f belongs to $L(-\infty,\infty)$, G exists as the transform of g in some sense or other, and G is $L(-\infty,\infty)$. We are unable to use Theorem 35, since we do not know that g is the transform of G. We have, however, the following theorem.

THEOREM 121. *Let* $f(x)$ *be* $L(-\infty,\infty)$, $g(x)$ *integrable over any finite interval, and let*

$$G(x)=\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} g(t)e^{ixt}\,dt \tag{6.8.1}$$

for all x, *and let* $G(x)$ *be everywhere finite, and* $L(-\infty,\infty)$. *Then* (2.1.1) *holds.*

The convergence of (6.8.1) *may fail for a finite set of values of* x, *provided that* $g(x)\to 0$ *as* $x\to\pm\infty$.

For $g(x)$ is the transform of $G(x)$, by Theorem 120, and the result therefore follows from Theorem 35.

6.9. Another uniqueness theorem.

We shall next prove a uniqueness theorem of a different type, in which (6.4.1) is not necessarily convergent, but in which there is an additional condition.

THEOREM 122.† *Let $e^{-yt}a(t)$, $e^{-yt}b(t)$ belong to $L(0,\infty)$ for every positive y, and let*

$$U(x,y)=\int\limits_0^\infty \{a(t)\cos xt+b(t)\sin xt\}e^{-yt}\,dt$$

be bounded for $y>0$ and all x. Then

$$\lim_{y\to 0} U(x,y)=f(x)$$

exists and is finite for almost all values of x, and

$$a(t)=\frac{1}{\pi}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)f(x)\cos xt\,dx,$$

$$b(t)=\frac{1}{\pi}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)f(x)\sin xt\,dx,$$

for almost all values of t.

Let $$\Phi(z)=\int\limits_0^\infty \{a(t)-ib(t)\}e^{izt}\,dt$$

be the analytic function of which $U(x,y)$ is the real part, and let $\Psi(z)=\exp\{-\Phi(z)\}$. Then $|\Psi(z)|$ is bounded above and below. Hence, by Theorem 94, $\Psi(z)$ tends to a finite non-zero limit for almost all x, as $y\to 0$. Hence $\Phi(z)$ tends to a finite limit for almost all x, and hence so does $U(x,y)$. Let the limit of $U(x,y)$ be $f(x)$. Now for $y>0$, $\eta>0$

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\eta}{(\xi-x)^2+\eta^2}U(x,y)\,dx=\frac{1}{\pi}\int\limits_0^\infty e^{-yt}a(t)\,dt\int\limits_{-\infty}^{\infty}\frac{\eta\cos xt}{(\xi-x)^2+\eta^2}\,dx+$$

$$+\frac{1}{\pi}\int\limits_0^\infty e^{-yt}b(t)\,dt\int\limits_{-\infty}^{\infty}\frac{\eta\sin xt}{(\xi-x)^2+\eta^2}\,dx$$

$$=\int\limits_0^\infty e^{-yt}\{a(t)e^{-\eta t}\cos\xi t+b(t)e^{-\eta t}\sin\xi t\}\,dt=U(\xi,y+\eta),$$

the inversion being justified by absolute convergence. Making $y\to 0$,

$$U(\xi,\eta)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\eta}{(\xi-x)^2+\eta^2}f(x)\,dx\cdot$$

by dominated convergence.

---

† Vorblunsky (2).

By Theorem 14,

$$e^{-yt}a(t) = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|x|}{\lambda}\right)U(x,y)\cos xt\,dx$$

$$= \frac{1}{\pi}\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_{0}^{\lambda}d\xi\int_{-\xi}^{\xi}U(x,y)\cos xt\,dx \tag{6.9.1}$$

for any positive y, for almost all values of t. Now

$$\int_{-\xi}^{\xi}U(x,y)\cos xt\,dx = \frac{1}{\pi}\int_{-\xi}^{\xi}\cos xt\,dx\int_{-\infty}^{\infty}\frac{y}{(x-u)^2+y^2}f(u)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,du\int_{-\xi}^{\xi}\frac{y\cos xt}{(x-u)^2+y^2}\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\{f(u)+f(-u)\}\,du\int_{0}^{\xi}\left\{\frac{y}{(x-u)^2+y^2}+\frac{y}{(x+u)^2+y^2}\right\}\cos xt\,dx$$

$$= \frac{1}{\pi}\left(\int_{0}^{\xi}\int_{0}^{\infty}-\int_{0}^{\xi}\int_{\xi}^{\infty}+\int_{\xi}^{\infty}\int_{0}^{\xi}\right) = \frac{1}{\pi}(J_1+J_2+J_3).$$

Also

$$\int_{0}^{\infty}\left\{\frac{y}{(x-u)^2+y^2}+\frac{y}{(x+u)^2+y^2}\right\}\cos xt\,dx$$

$$= \int_{-\infty}^{\infty}\frac{y}{(x-u)^2+y^2}\cos xt\,dx = \pi e^{-yt}\cos ut.$$

Hence

$$J_1 = \pi e^{-yt}\int_{0}^{\xi}\{f(u)+f(-u)\}\cos ut\,du. \tag{6.9.2}$$

Again, by the second mean-value theorem,

$$\left|\int_{\xi}^{\infty}\frac{y\cos xt}{(x+u)^2+y^2}\,dx\right| \leqslant \frac{2}{t}\,\frac{y}{(u+\xi)^2+y^2},$$

$$\left|\int_{\xi}^{\infty}\frac{y\cos xt}{(x-u)^2+y^2}\,dx\right| \leqslant \frac{2}{t}\,\frac{y}{(u-\xi)^2+y^2},$$

and also

$$\leqslant \int_{-\infty}^{\infty}\frac{y\,dx}{(x-u)^2+y^2} = \pi.$$

Hence, if $|f(u)+f(-u)| \leqslant M$,

$$|J_2| \leqslant \frac{2M}{t}\int\limits_0^{\xi} \frac{y}{(x+\xi)^2+y^2}\,dx+\frac{2M}{t}\int\limits_0^{\xi-\sqrt{y}} \frac{y}{(\xi-x)^2+y^2}\,dx+M\pi\int\limits_{\xi-\sqrt{y}}^{\xi} dy$$

$$=\frac{2M}{t}\left(\arctan\frac{2\xi}{y}-\arctan\frac{\xi}{y}\right)+\frac{2M}{t}\left(\arctan\frac{1}{\sqrt{y}}-\arctan\frac{\xi}{y}\right)+M\pi\sqrt{y}.$$

Also, if $u>\xi$,

$$\left|\int\limits_0^{\xi} \frac{y\cos xt}{(x+u)^2+t^2}\,dx\right| \leqslant \frac{2}{t}\,\frac{y}{u^2+y^2},$$

$$\left|\int\limits_0^{\xi} \frac{y\cos xt}{(x-u)^2+t^2}\,dx\right| \leqslant \frac{2}{t}\,\frac{y}{(u-\xi)^2+y^2}$$

and also $\leqslant \pi$

as before. Hence

$$|J_3| \leqslant \frac{2M}{t}\int\limits_{\xi}^{\infty} \frac{y}{u^2+y^2}\,du+\frac{2M}{t}\int\limits_{\xi+\sqrt{y}}^{\infty} \frac{y}{(u-\xi)^2+y^2}\,du+M\pi\int\limits_{\xi}^{\xi+\sqrt{y}} du$$

$$=\frac{2M}{t}\left(\frac{\pi}{2}-\arctan\frac{\xi}{y}\right)+\frac{2M}{t}\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{y}}\right)+M\pi\sqrt{y}.$$

Let $y_1, y_2,\ldots$ be a sequence of values of y tending to 0, and let E be the set of values of t for which (6.9.1) fails for any of these values of y. Then E is of measure 0. Let t be a point not in E. Then we can choose $y = y_n$ so small that the contribution of J_2 and J_3 to (6.9.1) is $<\epsilon$ for all $\lambda > 1$. Having fixed y, it follows from (6.9.2) that

$$\left|e^{-yt}a(t)-e^{-yt}\frac{1}{\pi\lambda}\int\limits_0^{\lambda} d\xi\int\limits_0^{\xi} \{f(u)+f(-u)\}\cos ut\,du\right| < 2\epsilon$$

for λ sufficiently large. Hence

$$a(t)=\lim_{\lambda\to\infty}\frac{1}{\pi\lambda}\int\limits_0^{\lambda} d\xi\int\limits_0^{\xi} \{f(u)+f(-u)\}\cos ut\,du.$$

Similarly we can prove the corresponding result for $b(t)$.

6.10. Special properties of transforms.

In this section we consider some special properties of sine and cosine transforms.

THEOREM 123. *Let $f(x)$ be non-increasing over* $(0,\infty)$, *integrable over* $(0,1)$, *and let $f(x)\to 0$ as $x\to\infty$. Then $F_s(x) \geqslant 0$.*

For

$$F_s(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^{\to\infty} f(y)\sin xy\, dy = \sqrt{\left(\frac{2}{\pi}\right)} \sum_{n=0}^{\infty} \int\limits_{n\pi/x}^{(n+1)\pi/x} f(y)\sin xy\, dy.$$

This is a series of alternately positive and negative terms, non-increasing in absolute value. Its sum is therefore $\geqslant 0$.

THEOREM 124. *Let $f(x)$ be a bounded function, which decreases steadily to 0 as $x \to \infty$ and is convex downwards. Then $F_c(x)$ is positive and belongs to $L(0,\infty)$.*

The conditions imply that $f(x)$ is the integral of $f'(x)$, which is negative and non-decreasing, and tends to a limit at infinity; and the limit is 0 since $f(x)$ is bounded. We can now integrate Fourier's cosine integral by parts, and obtain

$$F_c(x) = -\sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{x} \int\limits_0^{\infty} f'(y)\sin xy\, dy,$$

and this is positive, by the previous theorem. Also we may now take $x = 0$ in the analysis of Theorem 6, and obtain

$$\sqrt{\left(\frac{2}{\pi}\right)} \int\limits_{\to 0}^{\to\infty} F_c(u)\, du = \tfrac{1}{2}f(+0).$$

Hence $F_c(x)$ belongs to $L(0,\infty)$.

Neither of these theorems is true for transforms of the opposite kind. If $f(x) = 1$ $(0 \leqslant x \leqslant 1)$, 0 $(x > 1)$, then

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin x}{x},$$

which takes both signs. If $f(x) = e^{-x}$, then

$$F_s(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{x}{1+x^2},$$

which is positive but does not belong to $L(0,\infty)$. In fact if $F_s(x)$ belongs to $L(0,\infty)$, $f(x) \to 0$ both as $x \to 0$ and as $x \to \infty$, and so cannot be monotonic.

6.11. It is not quite easy to construct a monotonic $f(x)$ for which $F_c(x)$ is *not* integrable† over $(0,\infty)$. To do so, we first prove that *there is a function*

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x$$

---

† For a similar result for series see Szidon (1).

with $b_n \geqslant 0$, $\sum\limits_1^\infty b_n$ *convergent, and*

$$\int\limits_0^1 \frac{|\phi(x)|}{x}\,dx$$

divergent.

We have

$$\int\limits_{1/\lambda_n}^1 \frac{|\phi(x)|}{x}\,dx \geqslant \int\limits_{1/\lambda_n}^1 b_n|\sin\lambda_n x|\,\frac{dx}{x} - \int\limits_{1/\lambda_n}^1 \left|\sum_{\nu=1}^{n-1} b_\nu \sin\lambda_\nu x\right| \frac{dx}{x} - $$

$$- \int\limits_{1/\lambda_n}^1 \left|\sum_{\nu=n+1}^{\infty} b_\nu \sin\lambda_\nu x\right| \frac{dx}{x} = J_1 - J_2 - J_3.$$

Now

$$J_1 = b_n \int\limits_1^{\lambda_n} \left|\frac{\sin u}{u}\right| du > Ab_n \log\lambda_n,$$

$$J_2 < \int\limits_0^1 \sum_{\nu=1}^{n-1} b_\nu \lambda_\nu\,dx = \sum_{\nu=1}^{n-1} b_\nu \lambda_\nu,$$

$$J_3 < \int\limits_{1/\lambda_n}^1 \sum_{\nu=n+1}^{\infty} b_\nu \frac{dx}{x} = \log\lambda_n \sum_{\nu=n+1}^{\infty} b_\nu.$$

Hence $J_3 < \frac{1}{2}J_1$ if $b_\nu = k^{-\nu}$ with a sufficiently large k; and $J_1 \to \infty$, $J_2 = o(J_1)$ if e.g. $\lambda_1 = 1$, and

$$\lambda_n = 2^{(\lambda_1+\dots+\lambda_{n-1})/b_n}.$$

Then

$$\int\limits_{1/\lambda_n}^1 \frac{|\phi(x)|}{x}\,dx \to \infty,$$

the required result.

Theorem 125. *There is a function* $f(x)$, *continuous and steadily decreasing to* 0 *as* $x \to \infty$, *such that* $F_c(x)$ *does not belong to* $L(0,\infty)$.

We first obtain the result for a non-increasing function. Let $f(x) = c_n$ in $(a_{n-1}+\delta,\, a_n-\delta)$, where $c_n \to 0$ steadily and $a_n \to \infty$ steadily, and $0 < \delta < 1$; and let $f(x)$ be continuous and linear in the remaining intervals. Then, taking $a_0+\delta = 0$,

$$\sqrt{(\tfrac{1}{2}\pi)}F_c(x) = \sum_{n=1}^{\infty} c_n \int\limits_{a_{n-1}+\delta}^{a_n-\delta} \cos xt\,dt +$$

$$+ \sum_{n=1}^{\infty} \int\limits_{a_n-\delta}^{a_n+\delta} \left\{\tfrac{1}{2}(c_n+c_{n+1}) + \frac{(c_n-c_{n+1})(a_n-t)}{2\delta}\right\} \cos xt\,dt$$

$$= \sum_{n=1}^{\infty} c_n \frac{\sin(a_n-\delta)x - \sin(a_{n-1}+\delta)x}{x} +$$

$$+ \sum_{n=1}^{\infty} \left[c_{n+1} \frac{\sin(a_n+\delta)x}{x} - c_n \frac{\sin(a_n-\delta)x}{x} + \right.$$

$$\left. + \frac{c_n - c_{n+1}}{2\delta x^2} \{\cos(a_n-\delta)x - \cos(a_n+\delta)x\} \right]$$

$$= \frac{\sin \delta x}{\delta x^2} \sum_{n=1}^{\infty} (c_n - c_{n+1}) \sin a_n x.$$

By the lemma, we can choose c_n and a_n so that

$$\int_0^1 \left| \sum_{n=1}^{\infty} (c_n - c_{n+1}) \sin a_n x \right| \frac{dx}{x}$$

is divergent, and then so is $\int_0^1 |F_c(x)|\, dx$, since $\sin \delta x/(\delta x) \to 1$ as $x \to 0$.

We can plainly construct a steadily decreasing function $g(x)$, having derivatives of as many orders as we please, such that $f(x)-g(x)$ belongs to $L(0,\infty)$. Then the cosine transform of $f(x)-g(x)$ is bounded. Hence the cosine transform of $g(x)$ does not belong to $L(0,\infty)$.

6.12. Under special conditions $F_c(x)$ and $F_s(x)$ behave asymptotically like a power of x, either as $x \to 0$ or as $x \to \infty$, or both. The two simplest theorems of this character are as follows.†

THEOREM 126. *Let* $f(x) = x^{-\alpha}\phi(x)$, *where* $0 < \alpha < 1$, *and* $\phi(x)$ *is of bounded variation in* $(0, \infty)$. *Then*

$$F_c(x) \sim \phi(+0) \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(1-\alpha) \sin \tfrac{1}{2}\pi\alpha\, x^{\alpha-1} \quad (x \to \infty),$$

$$F_c(x) \sim \phi(\infty) \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(1-\alpha) \sin \tfrac{1}{2}\pi\alpha\, x^{\alpha-1} \quad (x \to 0).$$

$F_s(x)$ *satisfies similar conditions with* $\sin \frac{1}{2}\pi\alpha$ *replaced by* $\cos \frac{1}{2}\pi\alpha$.

We may suppose that $\phi(x)$ is positive non-increasing in $(0,\infty)$. Take the case $x \to \infty$. We have

$$\sqrt{(\tfrac{1}{2}\pi)} F_c(x) = \int_0^{\infty} t^{-\alpha}\phi(t) \cos xt\, dt$$

---

† Titchmarsh (9).

$$= x^{\alpha-1}\int_0^\infty u^{-\alpha}\phi(u/x)\cos u\,du$$

$$= x^{\alpha-1}\left(\int_0^\Delta+\int_\Delta^\infty\right) = x^{\alpha-1}(I_1+I_2).$$

By the second mean-value theorem

$$I_2 = \phi\left(\frac{\Delta}{x}\right)\int_\Delta^{\Delta'} u^{-\alpha}\cos u\,du = O(\Delta^{-\alpha})$$

uniformly with respect to x. Now, for a fixed Δ,

$$\int_0^\Delta \{\phi(+0)-\phi(u/x)\}u^{-\alpha}\cos u\,du = \{\phi(+0)-\phi(\Delta/x)\}\int_\delta^\Delta u^{-\alpha}\cos u\,du$$

$$= O\{\phi(+0)-\phi(\Delta/x)\} = o(1),$$

and

$$\phi(+0)\int_0^\Delta u^{-\alpha}\cos u\,du \to \phi(+0)\int_0^\infty u^{-\alpha}\cos u\,du = \phi(+0)\Gamma(1-\alpha)\sin\tfrac{1}{2}\pi\alpha.$$

Hence the result. Similarly in the case $x \to 0$.

Theorem 127. *Let $f(x)$ and $f'(x)$ be integrable over any finite interval not ending at $x = 0$; let $x^{\alpha+1}f'(x)$ be bounded for all x, and let $f(x) \sim x^{-\alpha}$ as $x \to \infty$ $(x \to 0)$. Then*

$$F_c(x) \sim \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(1-\alpha)\sin\tfrac{1}{2}\pi\alpha\, x^{\alpha-1}$$

as $x \to 0$ $(x \to \infty)$. $F_s(x)$ satisfies similar conditions with $\sin\frac{1}{2}\pi\alpha$ *replaced by* $\cos\frac{1}{2}\pi\alpha$.

Consider the case $x \to \infty$. We have

$$\sqrt{(\tfrac{1}{2}\pi)}F_c(x) = \int_0^\infty f(t)\cos xt\,dt = \int_0^{\Delta/x}+\int_{\Delta/x}^\infty = I_1+I_2.$$

Then

$$I_2 = -f\left(\frac{\Delta}{x}\right)\frac{\sin\Delta}{x}-\frac{1}{x}\int_{\Delta/x}^\infty f'(t)\sin xt\,dt$$

$$= O(\Delta^{-\alpha}x^{\alpha-1})+\frac{1}{x}\int_{\Delta/x}^\infty O(t^{-\alpha-1})\,dt$$

$$= O(\Delta^{-\alpha}x^{\alpha-1}).$$

Let $m(\xi) = \max\limits_{x \leqslant \xi} |x^{\alpha} f(x) - 1|$, so that $m(\xi) \to 0$ as $\xi \to 0$. Then

$$I_1 = \int\limits_0^{\Delta/x} t^{-\alpha} \cos xt\, dt + \int\limits_0^{\Delta/x} \{t^{\alpha} f(t) - 1\} t^{-\alpha} \cos xt\, dt$$

$$= x^{\alpha-1} \int\limits_0^{\Delta} u^{-\alpha} \cos u\, du + O\left\{m\left(\frac{\Delta}{x}\right)\left(\frac{\Delta}{x}\right)^{1-\alpha}\right\}$$

$$= x^{\alpha-1} \int\limits_0^{\infty} u^{-\alpha} \cos u\, du + O(x^{\alpha-1}\Delta^{-\alpha}) + O\left\{m\left(\frac{\Delta}{x}\right)\left(\frac{\Delta}{x}\right)^{1-\alpha}\right\},$$

and the result follows on choosing Δ large enough, and then x large enough.

Similarly if $x \to 0$.

6.13. Order of magnitude of transforms.

There are various more or less trivial results; if $(1+|x|^n)f(x)$ belongs to $L(-\infty,\infty)$, the equation

$$F(x) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{\infty} f(t) e^{ixt}\, dt$$

can be differentiated n times. It follows that $F(x), F'(x), \ldots, F^{(n)}(x)$ are all continuous and tend to 0 at infinity.

If $f(x), \ldots, f^{(n-1)}(x)$ are continuous and tend to 0 at infinity, and $f^{(n)}(x)$ is L, then by repeated integration by parts

$$F(x) = \left(\frac{i}{x}\right)^n \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{\infty} f^{(n)}(t) e^{ixt}\, dt.$$

Hence $x^n F(x) \to 0$ as $x \to \pm\infty$.

Similarly, if $(1+|x|^n)f(x)$ belongs to $L^2(-\infty,\infty)$, then

$$F(x), \ldots, F^{(n-1)}(x)$$

are continuous and tend to 0 at infinity, and $F^{(n)}(x)$ is $L^2(-\infty,\infty)$; and conversely.

Other results have been given in Theorem 26.

The idea underlying the following theorem is that *both a function and its transform cannot be too small at infinity.*† The result is

THEOREM 128. *Let $f(x)$ and $F_c(x)$ be Fourier cosine transforms, and let each be $O(e^{-\frac{1}{2}x^2})$ as $x \to \infty$. Then*

$$f(x) = F_c(x) = Ce^{-\frac{1}{2}x^2}.$$

---

† Hardy (19), Ingham (1), Morgan (1), Paley and Wiener, *Fourier Transforms*, § 19.

We use the following lemma.

LEMMA. *Let $\phi(z)$ be an integral function, $\phi(z) = O(e^{a|z|})$ for all z, and $\phi(x) = O(e^{-ax})$ for real positive $x \to \infty$, a being a positive constant. Then $\phi(z) = Ce^{-az}$.*

There is a constant C such that

$$|\phi(x)| \leqslant Ce^{-ax}, \qquad |\phi(re^{i\alpha})| \leqslant Ce^{ar},$$

where $0 < \alpha < \pi$. Hence, by a theorem of Phragmén and Lindelöf,†

$$|\phi(re^{i\theta})| \leqslant Ce^{rH(\theta)} \quad (0 \leqslant \theta \leqslant \alpha),$$

where
$$H(\theta) = \frac{-a\sin(\alpha-\theta)+a\sin\theta}{\sin\alpha} = a\frac{\sin(\theta-\frac{1}{2}\alpha)}{\sin\frac{1}{2}\alpha}.$$

Here we can keep θ fixed and make $\alpha \to \pi$. Then $H(\theta) \to -a\cos\theta$, and it follows that

$$|\phi(z)| \leqslant Ce^{-ar\cos\theta} \quad (0 \leqslant \theta < \pi).$$

Similarly, we obtain the same inequality for $-\pi < \theta \leqslant 0$; and also, by continuity, for $\theta = \pi$. Hence $e^{az}\phi(z)$ is a bounded integral function, and so is a constant.

To prove the theorem we have

$$F_c(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(t)\cos zt\,dt.$$

By the condition on $f(t)$ this is an even integral function of z; and if $|z| = r$,

$$|F_c(z)| \leqslant K \int_0^\infty e^{-\frac{1}{2}t^2}\cosh rt\,dt = Ke^{\frac{1}{2}r^2}.$$

Hence $F_c(\sqrt{z})$ is an integral function which satisfies the conditions of the lemma, with $a = \frac{1}{2}$. Hence

$$F_c(\sqrt{z}) = Ce^{-\frac{1}{2}z},$$
$$F_c(z) = Ce^{-\frac{1}{2}z^2},$$

and also the result for $f(z)$ follows, by a familiar formula.

More general results can be obtained in a similar way. Suppose for example that

$$f(x) = O(x^{2k}e^{-\frac{1}{2}x^2}), \qquad F_c(x) = O(x^{2k}e^{-\frac{1}{2}x^2}),$$

---

† The argument is that of Titchmarsh, *Theory of Functions*, § 5.71, with $\delta = 0$.

where k is an integer. Then $F_c(z)$ is an even integral function; and

$$|F_c(z)| < K\int_0^\infty e^{-\frac{1}{2}t^2}t^{2k}\cosh rt\,dt$$

$$= K\left(\frac{d}{dt}\right)^{2k}\int_0^\infty e^{-\frac{1}{2}t^2}\cosh rt\,dt$$

$$= K\left(\frac{d}{dt}\right)^{2k}e^{\frac{1}{2}r^2} = O(r^{2k}e^{\frac{1}{2}r^2}).$$

Let $\phi(z) = F_c(\surd z)$. Then $\phi(z)$ is an integral function; and, if $a_0, \ldots, a_{k-1}$ are properly chosen, so is

$$\psi(z) = z^{-2k}\{\phi(z)-(a_0+a_1 z+\ldots+a_{k-1}z^{k-1})e^{-\frac{1}{2}z}\}.$$

Hence $\psi(z)$ satisfies the conditions of the lemma, and so is $Ce^{-\frac{1}{2}z}$. Hence

$$F_c(z) = (a_0+a_1 z^2+\ldots+a_k z^{2k})e^{-\frac{1}{2}z^2},$$

and $f(z)$ is another expression of the same form.

VII

EXAMPLES AND APPLICATIONS[†]

7.1. Cosine transforms. SIMPLE pairs of cosine transforms are

$$1\ (0,a),\quad 0\ (a,\infty),\qquad \sqrt{\left(\frac{2}{\pi}\right)}\frac{\sin ax}{x}, \tag{7.1.1}$$

$$\cos x\ (0,a),\quad 0\ (a,\infty),\qquad \frac{1}{\sqrt{(2\pi)}}\left\{\frac{\sin a(1-x)}{1-x}+\frac{\sin a(1+x)}{1+x}\right\}, \tag{7.1.2}$$

$$\sin x\ (0,a),\quad 0\ (a,\infty),\qquad \frac{1}{\sqrt{(2\pi)}}\left\{\frac{1-\cos a(1-x)}{1-x}+\frac{1-\cos a(1+x)}{1+x}\right\}, \tag{7.1.3}$$

$$e^{-x},\qquad \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{1+x^2}. \tag{7.1.4}$$

Generally, the cosine transform of any even rational function, regular on the real axis and $O(1/x^2)$ at infinity, can be evaluated by contour integration; for example

$$\frac{1}{1+x^4},\qquad \sqrt{\left(\frac{\pi}{2}\right)}e^{-x/\sqrt{2}}\sin\left(\frac{x}{\sqrt{2}}+\frac{\pi}{4}\right). \tag{7.1.5}$$

Another familiar process of contour integration gives the pair

$$\frac{1}{\cosh \pi x},\qquad \frac{1}{\sqrt{(2\pi)}\cosh\frac{1}{2}x}. \tag{7.1.6}$$

Next

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty e^{-\frac{1}{2}x^2}\cos xu\,dx$$

$$=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+ixu}\,dx=\frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}u^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-iu)^2}\,dx$$

$$=\frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}u^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi^2}\,d\xi=Ce^{-\frac{1}{2}u^2}$$

(by Cauchy's theorem). The cosine formula then gives $C^2=1$, whence $C=1$ since $C>0$. Hence we have the pair

$$e^{-\frac{1}{2}x^2},\qquad e^{-\frac{1}{2}x^2}. \tag{7.1.7}$$

All the above examples belong to obvious L-classes.

[†] An extensive list of Fourier transforms is given by Campbell and Foster, *Fourier Integrals for Practical Applications*.

Next we have

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty e^{-\frac{1}{2}\lambda y^2}\cos xy\,dy = \sqrt{\left(\frac{2}{\pi\lambda}\right)}\int_0^\infty e^{-\frac{1}{2}y^2}\cos\frac{xy}{\sqrt{\lambda}}\,dy$$
$$= \frac{1}{\sqrt{\lambda}}e^{-\frac{1}{2}x^2/\lambda}$$

by the formula just proved.

This is true primarily for real positive λ, but it can be extended by the theory of analytic continuation to all values of λ with $\mathbf{R}(\lambda) \geqslant 0$. Taking $\lambda = i$,

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty e^{-\frac{1}{2}iy^2}\cos xy\,dy = e^{\frac{1}{2}ix^2-\frac{1}{4}i\pi}.$$

Taking real and imaginary parts. we obtain the transforms

$$\cos\tfrac{1}{2}x^2, \qquad \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}x^2+\sin\tfrac{1}{2}x^2), \tag{7.1.8}$$

$$\sin\tfrac{1}{2}x^2, \qquad \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}x^2-\sin\tfrac{1}{2}x^2). \tag{7.1.9}$$

The Fourier formulae arising from these give examples of Theorem 11, case (i).

We define the Bessel function of order ν by

$$J_\nu(x) = \sum_{n=0}^\infty \frac{(-1)^n(\frac{1}{2}x)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \quad (\nu > -1). \tag{7.1.10}$$

Then[†]

$$\int_0^1 (1-y^2)^{\nu-\frac{1}{2}}\cos xy\,dy = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}\int_0^1 (1-y^2)^{\nu-\frac{1}{2}}y^{2n}\,dy$$
$$= \frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(\nu+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2n)!\,\Gamma(\nu+n+1)}x^{2n}$$
$$= \tfrac{1}{2}\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{2^{2n}n!\,\Gamma(\nu+n+1)}.$$

Hence we have the cosine transforms

$$(1-x^2)^{\nu-\frac{1}{2}}\ (0<x<1),\quad 0\ (x>1), \qquad 2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})x^{-\nu}J_\nu(x). \tag{7.1.11}$$

[†] Watson, *Theory of Bessel Functions*, § 3.3 (2). This work is referred to later as 'Watson'.

These belong to L^2 if $\nu > 0$, and to L^p, $L^{p'}$ respectively if $1/p > \frac{1}{2}-\nu$, $\nu > -\frac{1}{2}$.

In each case we can introduce a parameter, since the transform of $f(\lambda x)$ is

$$\frac{1}{\lambda}F_c\left(\frac{x}{\lambda}\right),$$

and similarly for sine transforms.

7.2. Sine transforms. A simple pair is

$$e^{-x}, \qquad \sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{1+x^2}, \tag{7.2.1}$$

and, generally, the sine transform of any odd rational function, regular on the real axis and $O(1/x)$ at infinity, can be evaluated by contour integration.

Other familiar methods of contour integration give the pairs

$$\frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}, \qquad \frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}, \tag{7.2.2}$$

$$\frac{1}{\sinh x\sqrt{(\frac{1}{2}\pi)}}-\frac{1}{x\sqrt{(\frac{1}{2}\pi)}}, \qquad \tanh\{x\sqrt{(\tfrac{1}{2}\pi)}\}-1. \tag{7.2.3}$$

The pair

$$xe^{-\frac{1}{2}x^2}, \qquad xe^{-\frac{1}{2}x^2} \tag{7.2.4}$$

may be obtained by differentiation from (7.1.7). Next†

$$\int_0^\infty \frac{\sin y}{y}\sin xy\,dy = \frac{1}{2}\int_0^\infty \frac{\cos(1-x)y-\cos(1+x)y}{y}\,dy = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|.$$

Hence we obtain the pair

$$\frac{\sin x}{x}, \qquad \frac{1}{\sqrt{(2\pi)}}\log\left|\frac{1+x}{1-x}\right|. \tag{7.2.5}$$

If $\nu > \frac{1}{2}$ we obtain by partial integration from (7.1.11) the pair

$$x(1-x^2)^{\nu-\frac{3}{2}}\ (0 < x < 1), \quad 0\ (x > 1), \qquad 2^{\nu-\frac{3}{2}}\Gamma(\nu-\tfrac{1}{2})x^{1-\nu}J_\nu(x). \tag{7.2.6}$$

We define the Struve's function‡ of order ν by

$$\mathbf{H}_\nu(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}x)^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})} \quad (\nu > -\tfrac{3}{2}). \tag{7.2.7}$$

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† e.g. as in § 5.2.

Watson, § 10.4.

Then

$$\int_0^1 (1-y^2)^{\nu-\frac{1}{2}}\sin xy\,dy = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\int_0^1 (1-y^2)^{\nu-\frac{1}{2}}y^{2n+1}\,dy$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\nu+\frac{1}{2})n!}{(2n+1)!\,\Gamma(\nu+n+\frac{3}{2})}x^{2n+1}$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}.$$

Hence we have the sine transforms

$$(1-x^2)^{\nu-\frac{1}{2}}\ (0 < x < 1),\quad 0\ (x > 1),\qquad 2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})x^{-\nu}\mathbf{H}_\nu(x). \tag{7.2.8}$$

7.3. The Parseval formulae. We obtain simple examples of (2.1.4) or (2.1.6) by taking f and g rational functions; for example, let $f(x) = 1/(x^2+a^2)$, $F_c(x) = \sqrt{(\frac{1}{2}\pi)}e^{-ax}/a$, and similarly g, G_c, with b for a. We obtain

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab}\int_0^{\infty}e^{-ax-bx}\,dx = \frac{\pi}{2ab(a+b)}. \tag{7.3.1}$$

As another type, let $f(x) = 1$ $(0 < x < a)$, 0 $(x \geqslant a)$, $F_c(x) = \sqrt{(2/\pi)}\sin ax/x$, and similarly g, G_c, with b for a. We obtain

$$\int_0^{\infty}\frac{\sin ax\sin bx}{x^2}\,dx = \tfrac{1}{2}\pi\int_0^{\min(a,b)}dx = \tfrac{1}{2}\pi\min(a,b). \tag{7.3.2}$$

Similarly, from (2.1.6) and (7.2.5) we obtain

$$\int_0^{\infty}\log\left|\frac{a+x}{a-x}\right|\log\left|\frac{b+x}{b-x}\right|dx = 2\pi\int_0^{\infty}\frac{\sin ax\sin bx}{x^2}\,dx = \pi^2\min(a,b), \tag{7.3.3}$$

by (7.3.2); or we can obtain this directly from Theorem 91. All the above formulae come under Theorem 52.

We can deduce some of the familiar Γ-function formulae from Parseval's formula.[†] Define $\Gamma(a)$ by

$$\Gamma(a) = \int_0^{\infty}e^{-x}x^{a-1}\,dx\quad (a > 0), \tag{7.3.4}$$

[†] See Hardy (3).

and let

$$c(a) = \int_0^\infty x^{a-1}\cos x\,dx \quad (0 < a < 1), \tag{7.3.5}$$

$$s(a) = \int_0^\infty x^{a-1}\sin x\,dx \quad (-1 < a < 1). \tag{7.3.6}$$

Then the cosine transform of x^{a-1} is

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty y^{a-1}\cos xy\,dy = x^{-a}\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty u^{a-1}\cos u\,du = x^{-a}\sqrt{\left(\frac{2}{\pi}\right)}c(a).$$

Similarly, the sine transform of x^{a-1} is $x^{-a}\sqrt{\left(\frac{2}{\pi}\right)}s(a)$.

We can prove, by contour integration or term-by-term integration of series, that

$$\int_0^\infty \frac{x^{-a}}{1+x^2}\,dx = \tfrac{1}{2}\pi\sec\tfrac{1}{2}a\pi \quad (-1 < a < 1). \tag{7.3.7}$$

In (2.1.4) let $f(x) = e^{-x}$, $g(x) = x^{a-1}$. We obtain

$$\Gamma(a) = \frac{2}{\pi}\int_0^\infty \frac{x^{-a}c(a)}{1+x^2}\,dx = c(a)\sec\tfrac{1}{2}a\pi \quad (0 < a < 1), \tag{7.3.8}$$

e.g. by Theorem 36. Similarly, by (2.1.6),

$$\Gamma(a) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}x^{-a}s(a)\,dx = s(a)\operatorname{cosec}\tfrac{1}{2}a\pi \quad (0 < a < 1). \tag{7.3.9}$$

Also, by the above rule, the cosine transform of $x^{-a}\sqrt{\left(\frac{2}{\pi}\right)}c(a)$ is

$$x^{a-1}\frac{2}{\pi}c(a)c(1-a);$$

since it is also x^{a-1}, by Theorem 6, it follows that

$$c(a)c(1-a) = \tfrac{1}{2}\pi \quad (0 < a < 1). \tag{7.3.10}$$

Similarly,

$$s(a)s(1-a) = \tfrac{1}{2}\pi \quad (0 < a < 1). \tag{7.3.11}$$

Also, (7.3.8) and (7.3.10) give

$$\Gamma(a)\Gamma(1-a) = \frac{c(a)c(1-a)}{\frac{1}{2}\sin a\pi} = \frac{\pi}{\sin a\pi}. \tag{7.3.12}$$

In particular

$$c(\tfrac{1}{2}) = \sqrt{(\tfrac{1}{2}\pi)}, \qquad s(\tfrac{1}{2}) = \sqrt{(\tfrac{1}{2}\pi)}, \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{7.3.13}$$

We have also obtained the cosine transforms

$$x^{a-1},\qquad \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(a)\cos\tfrac{1}{2}a\pi\, x^{-a}\quad (0<a<1) \tag{7.3.14}$$

and the sine transforms

$$x^{a-1},\qquad \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(a)\sin\tfrac{1}{2}a\pi\, x^{-a}\quad (0<a<1). \tag{7.3.15}$$

7.4. Some Bessel-function examples. From (2.1.4), (7.1.11), and (7.3.14) we obtain

$$\begin{aligned}\int\limits_0^\infty J_\nu(x)x^{a-\nu-1}\,dx &= \sqrt{\left(\frac{2}{\pi}\right)}\frac{\Gamma(a)\cos\frac{1}{2}a\pi}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int\limits_0^1 (1-x^2)^{\nu-\frac{1}{2}}x^{-a}\,dx\\ &= \frac{\Gamma(a)\cos\frac{1}{2}a\pi}{2^{\nu-1}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\,\frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}a)}{2\Gamma(\nu-\frac{1}{2}a+1)}\\ &= \frac{2^{a-\nu-1}\Gamma(\frac{1}{2}a)}{\Gamma(\nu-\frac{1}{2}a+1)}.\end{aligned} \tag{7.4.1}$$†

This is a case of Theorem 36 if $\nu>-\frac{1}{2}$, $0<a<1$ (taking $f(x)=(1-x^2)^{\nu-\frac{1}{2}}$ $(0<x<1)$, 0 $(x>1)$, and $g(x)=x^{-a}$). Actually the integral converges if $0<a<\nu+\frac{3}{2}$, so that the result holds by analytic continuation in this wider range.

Similarly, from (2.1.6), (7.2.8), and (7.3.15) we obtain

$$\begin{aligned}\int\limits_0^\infty \mathbf{H}_\nu(x)x^{a-\nu-1}\,dx &= \sqrt{\left(\frac{2}{\pi}\right)}\frac{\Gamma(a)\sin\frac{1}{2}a\pi}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int\limits_0^1 (1-x^2)^{\nu-\frac{1}{2}}x^{-a}\,dx\\ &= \frac{2^{a-\nu-1}\Gamma(\frac{1}{2}a)\tan\frac{1}{2}a\pi}{\Gamma(\nu-\frac{1}{2}a+1)}\quad (-1<a<\nu+\tfrac{3}{2}).\end{aligned} \tag{7.4.2}$$‡

As an example of (2.1.8) let $-\frac{1}{2}<\nu<0$,

$$f(x)=\sqrt{(2/\pi)}\Gamma(2\nu+1)\cos\nu\pi|x|^{-2\nu-1}\operatorname{sgn}x,\qquad F(x)=i|x|^{2\nu}\operatorname{sgn}x$$

by (7.3.15), and

$$g(x)=2^{\frac{1}{2}-\nu}(1-x^2)^{\nu-\frac{1}{2}}/\Gamma(\nu+\tfrac{1}{2})\quad (|x|<1),\qquad 0\quad (|x|>1),$$
$$G(x)=|x|^{-\nu}J_\nu(|x|),$$

by (7.1.11). Then if $x\neq 1$,

$$\begin{aligned}&2\int\limits_0^{\to\infty} t^\nu J_\nu(t)\sin xt\,dt\\ &\quad=\sqrt{\left(\frac{2}{\pi}\right)}\frac{\Gamma(2\nu+1)\cos\nu\pi}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int\limits_{-1}^1 (1-u^2)^{\nu-\frac{1}{2}}|x-u|^{-2\nu-1}\operatorname{sgn}(x-u)\,du.\end{aligned} \tag{7.4.3}$$

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† Watson, § 13.24 (1).

‡ Ibid., § 13.24 (2).

This may be justified by Theorem 39.

If $x > 1$, the integral on the right-hand side is†

$$\int_{-1}^{1} (1-u^2)^{\nu-\frac{1}{2}}(x-u)^{-2\nu-1}\,du = \frac{\{\Gamma(\nu+\frac{1}{2})\}^2}{\Gamma(2\nu+1)}\frac{2^{2\nu}}{(x^2-1)^{\nu+\frac{1}{2}}}.$$

If $0 < x < 1$, the integral is 0; for consider

$$\int (1-w^2)^{\nu-\frac{1}{2}}(x-w)^{-2\nu-1}\,dw.$$

This integral, taken round a circle of centre the origin and radius R, tends to 0 as $R \to \infty$. On reducing the contour to the real axis from -1 to 1 described twice, and allowing for the change of value of the integrand at -1, x, and 1, we obtain a multiple of the above integral.

It follows that we have the sine transforms

$$x^\nu J_\nu(x), \qquad \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)(x^2-1)^{\nu+\frac{1}{2}}} \quad (x > 1), \quad 0 \quad (x < 1). \tag{7.4.4}$$‡

Actually (7.4.3) converges for $\nu < \frac{1}{2}$, and the result holds by analytic continuation in this wider range. The functions belong to L^p, $L^{p'}$ if

$$p > \frac{1}{\frac{1}{2}-\nu} \quad (\nu \geqslant 0); \qquad \frac{1}{\frac{1}{2}-\nu} < p < -\frac{1}{2\nu} \quad (-\tfrac{1}{2} < \nu < 0). \tag{7.4.5}$$

From (2.1.6), generalized as in (2.1.22), and (7.2.6) and (7.4.4), we obtain, if $0 < a < b$,

$$\int_0^\infty (ax)^\mu J_\mu(ax)(bx)^{1-\nu}J_\nu(bx)\,dx$$

$$= \frac{1}{ab}\int_a^b \frac{2^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu)}\left(\frac{x^2}{a^2}-1\right)^{-\mu-\frac{1}{2}} \frac{2^{\frac{3}{2}-\nu}}{\Gamma(\nu-\frac{1}{2})}\cdot\frac{x}{b}\left(1-\frac{x^2}{b^2}\right)^{\nu-\frac{3}{2}}\,dx,$$

while the left-hand side is 0 if $0 < b \leqslant a$. Hence

$$\int_0^\infty x^{\mu-\nu+1}J_\mu(ax)J_\nu(bx)\,dx$$

$$= \frac{2^{\mu-\nu+1}a^\mu(b^2-a^2)^{\nu-\mu-1}}{\Gamma(\nu-\mu)b^\nu} \quad (0 < a < b), \qquad 0 \quad (a \geqslant b). \tag{7.4.6}$$

The process is justified by L^2 theory if $-\frac{1}{4} < \mu < 0$, $\nu > 1$. As usual, the result holds in a wider range.

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† See Titchmarsh, *Theory of Functions*, p. 63, ex. 19.

‡ Watson, § 6.13 (3).

Next let

$$f(x) = \sin x/x, \qquad F(x) = \sqrt{(\tfrac{1}{2}\pi)} \ \ (|x| < 1), \quad 0 \ \ (|x| > 1),$$

and

$$g(x) = J_0(x) \quad (x > 0), \qquad 0 \quad (x < 0),$$

$$G(x) = \frac{1}{\sqrt{(2\pi)}} \int\limits_0^\infty J_0(y) \cos xy\, dy + \frac{i}{\sqrt{(2\pi)}} \int\limits_0^\infty J_0(y) \sin xy\, dy$$

$$= \frac{1}{\sqrt{\{2\pi(1-x^2)\}}} \quad (|x| < 1), \qquad \frac{i \operatorname{sgn} x}{\sqrt{\{2\pi(x^2-1)\}}} \quad (|x| > 1),$$

by (7.1.11) and (7.4.4). Then (2.1.8) gives

$$\int\limits_0^\infty J_0(t) \frac{\sin(x-t)}{x-t}\, dt = \frac{1}{2} \int\limits_{-1}^1 \frac{e^{-ixt}}{\sqrt{(1-t^2)}}\, dt = \int\limits_0^1 \frac{\cos xt}{\sqrt{(1-t^2)}}\, dt = \tfrac{1}{2}\pi J_0(x). \tag{7.4.7†}$$

Here $f(x)$ and $G(x)$ belong to L^p if $1 < p < 2$.

From (7.2.1), (7.4.4), and (2.1.6) we deduce

$$\int\limits_0^\infty e^{-ax} x^\nu J_\nu(x)\, dx = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_1^\infty \frac{x}{a^2+x^2} \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)(x^2-1)^{\nu+\frac{1}{2}}}\, dx$$

$$= \frac{2^\nu}{\sqrt{\pi}\, \Gamma(\frac{1}{2}-\nu)} \int\limits_0^1 \frac{u^{\nu-\frac{1}{2}}}{(a^2u+1)(1-u)^{\nu+\frac{1}{2}}}\, du = \frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(a^2+1)^{\nu+\frac{1}{2}}}. \tag{7.4.8‡}$$

Here the L^p theory applies if p satisfies (7.4.5). The result holds by analytic continuation if $\nu > -\frac{1}{2}$.

7.5. Some integrals of Ramanujan.[||] Let

$$\phi(x) = \int\limits_{-\infty}^\infty \frac{e^{-i\pi u^2}}{\cosh \pi u} e^{-ixu}\, du. \tag{7.5.1}$$

Then

$$\phi(x+i\pi) + \phi(x-i\pi) = 2 \int\limits_{-\infty}^\infty e^{-i\pi u^2 - ixu}\, du$$

$$= 2e^{\frac{ix^2}{4\pi} - \frac{i\pi}{4}}. \tag{7.5.2}$$

Again, by (7.1.8) and (7.1.9) the transform of $e^{-i\pi x^2}$ is $(2\pi)^{-\frac{1}{2}} e^{\frac{ix^2}{4\pi} - \frac{i\pi}{4}}$,

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[†] Watson, § 13.55 (4).

[‡] Ibid., § 13.2 (5).

[||] Ramanujan (2), (5), Watson (5).

and that of $\operatorname{sech}\pi x$ is $(2\pi)^{-\frac{1}{2}}\operatorname{sech}\frac{1}{2}x$. Hence, by (2.1.8),

$$\phi(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{\frac{i(x-u)^2}{4\pi}-\frac{i\pi}{4}}}{\cosh\frac{1}{2}u}\,du=e^{\frac{ix^2}{4\pi}-\frac{i\pi}{4}}\int\limits_{-\infty}^{\infty}\frac{e^{i\pi v^2-ixv}}{\cosh\pi v}\,dv.$$

This integral is of the same form as the original one, and we can repeat the process which gave (7.5.2). We obtain

$$e^{-\frac{i(x+i\pi)^2}{4\pi}}\phi(x+i\pi)+e^{-\frac{i(x-i\pi)^2}{4\pi}}\phi(x-i\pi)=2e^{-\frac{i\pi}{4}}\int\limits_{-\infty}^{\infty}e^{i\pi v^2-ixv}\,dv$$

$$=2e^{-\frac{ix^2}{4\pi}},$$

i.e.

$$e^{\frac{1}{2}x}\phi(x+i\pi)+e^{-\frac{1}{2}x}\phi(x-i\pi)=2e^{-\frac{i\pi}{4}}. \tag{7.5.3}$$

Eliminating $\phi(x-i\pi)$,

$$(e^{\frac{1}{2}x}-e^{-\frac{1}{2}x})\phi(x+i\pi)=2e^{-\frac{i\pi}{4}}\left(1-e^{\frac{ix^2}{4\pi}-\frac{1}{2}x}\right), \tag{7.5.4}$$

and, replacing x by $x-i\pi$, we obtain

$$\phi(x)=\frac{e^{\frac{i\pi}{4}}-ie^{\frac{ix^2}{4\pi}}}{\cosh\frac{1}{2}x}. \tag{7.5.5}$$

Taking real and imaginary parts, we obtain the formulae

$$\int\limits_{0}^{\infty}\frac{\cos\pi u^2\cos xu}{\cosh\pi u}\,du=\frac{\sin\dfrac{x^2}{4\pi}+\dfrac{1}{\sqrt{2}}}{2\cosh\frac{1}{2}x}, \tag{7.5.6}$$

$$\int\limits_{0}^{\infty}\frac{\sin\pi u^2\cos xu}{\cosh\pi u}\,du=\frac{\cos\dfrac{x^2}{4\pi}-\dfrac{1}{\sqrt{2}}}{2\cosh\frac{1}{2}x}. \tag{7.5.7}$$

Similar integrals with denominator $\sinh\pi u$ may be evaluated in a similar way, or deduced from the previous ones, as follows. We have

$$\phi(x+i\pi)=\int\limits_{-\infty}^{\infty}e^{-i\pi u^2-ixu}\frac{e^{\pi u}}{\cosh\pi u}\,du$$

$$=\int\limits_{-\infty}^{\infty}e^{-i\pi u^2-ixu}(1+\tanh\pi u)\,du$$

$$=e^{\frac{ix^2}{4\pi}-\frac{i\pi}{4}}-2i\int\limits_{0}^{\infty}e^{-i\pi u^2}\sin xu\tanh\pi u\,du. \tag{7.5.8}$$

Now by (7.5.4)

$$\phi(x+i\pi)-e^{\frac{ix^2}{4\pi}-\frac{i\pi}{4}} = e^{-\frac{i\pi}{4}}\left\{\frac{1-e^{\frac{ix^2}{4\pi}-\frac{1}{2}x}}{\sinh\frac{1}{2}x}-e^{\frac{ix^2}{4\pi}}\right\}$$

$$= \frac{e^{-\frac{i\pi}{4}}}{\sinh\frac{1}{2}x}\left(1-e^{\frac{ix^2}{4\pi}}\cosh\tfrac{1}{2}x\right).$$

Also $$\tanh\pi u = \frac{1}{\pi}\int\limits_0^\infty \frac{\sin uv}{\sinh\frac{1}{2}v}\,dv.$$

Hence the second term on the right of (7.5.8) is

$$-\frac{2i}{\pi}\int\limits_0^\infty e^{-i\pi u^2}\sin xu\,du\int\limits_0^\infty \frac{\sin uv}{\sinh\frac{1}{2}v}\,dv$$

$$= -\frac{2i}{\pi}\int\limits_0^\infty \frac{dv}{\sinh\frac{1}{2}v}\int\limits_0^\infty e^{-i\pi u^2}\sin xu\sin vu\,du$$

$$= -\frac{1}{\pi}e^{\frac{ix^2}{4\pi}-\frac{i\pi}{4}}\int\limits_0^\infty \frac{e^{\frac{iv^2}{4\pi}}\sin(xv/2\pi)}{\sinh\frac{1}{2}v}\,dv$$

$$= -2e^{\frac{ix^2}{4\pi}-\frac{i\pi}{4}}\int\limits_0^\infty \frac{e^{i\pi y^2}\sin xy}{\sinh\pi y}\,dy.$$

Hence $$\int\limits_0^\infty \frac{e^{i\pi y^2}\sin xy}{\sinh\pi y}\,dy = \frac{\cosh\frac{1}{2}x-e^{-\frac{ix^2}{4\pi}}}{2\sinh\frac{1}{2}x},$$

i.e. $$\int\limits_0^\infty \frac{\cos\pi y^2\sin xy}{\sinh\pi y}\,dy = \frac{\cosh\frac{1}{2}x-\cos(x^2/4\pi)}{2\sinh\frac{1}{2}x}, \qquad (7.5.9)$$

and $$\int\limits_0^\infty \frac{\sin\pi y^2\sin xy}{\sinh\pi y}\,dy = \frac{\sin(x^2/4\pi)}{2\sinh\frac{1}{2}x}. \qquad (7.5.10)$$

7.6. Some Γ-function formulae.† The formula

$$\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos t)^{a-2}e^{ixt}\,dt = \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma(\frac{1}{2}a+\frac{1}{2}x)\Gamma(\frac{1}{2}a-\frac{1}{2}x)} \quad (a>1) \qquad (7.6.1)$$

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† Ramanujan (4), (6).

may be obtained by calculating

$$\int \left(w+\frac{1}{w}\right)^{a-2} w^{x-1}\,dw$$

taken round the contour formed by joining the points $-i$, i by the imaginary axis, and by the right-hand half of the unit circle.

The reciprocal formula is

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \frac{e^{-ixt}}{\Gamma(\frac{1}{2}a+\frac{1}{2}x)\Gamma(\frac{1}{2}a-\frac{1}{2}x)}\,dx &= \frac{2^{a-1}(\cos t)^{a-2}}{\Gamma(a-1)} \quad (|t| < \tfrac{1}{2}\pi)\\ &= 0 \quad (|t| \geqslant \tfrac{1}{2}\pi),\end{aligned} \tag{7.6.2}$$

or, putting $a = \alpha+\beta$, $x = 2u+\alpha-\beta$, $t = \frac{1}{2}y$,

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \frac{e^{-iuy}}{\Gamma(\alpha+u)\Gamma(\beta-u)}\,du &= \frac{(2\cos\frac{1}{2}y)^{\alpha+\beta-2}e^{\frac{1}{2}iy(\alpha-\beta)}}{\Gamma(\alpha+\beta-1)} \quad (|y| < \pi)\\ &= 0 \quad (|y| \geqslant \pi).\end{aligned} \tag{7.6.3}$$

Here $\qquad F(x) = \{\Gamma(\alpha+u)\Gamma(\beta-u)\}^{-1} = O(|u|^{1-\alpha-\beta})$

as $u \to \pm\infty$. The functions $F(x), f(x)$, related by (7.6.3) both belong to L^p $(p \geqslant 1)$ if $\alpha+\beta > 2$; if $1 < \alpha+\beta \leqslant 2$ they belong to L^p, $L^{p'}$ respectively if $p(\alpha+\beta-1) > 1$. In the latter case (7.6.3) is non-absolutely convergent; this may be verified from the asymptotic expressions for the Γ-functions, or by Theorem 59 and its extension to L^p.

The particular case $y = 0$ is

$$\int\limits_{-\infty}^{\infty} \frac{du}{\Gamma(\alpha+u)\Gamma(\beta-u)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \quad (\alpha+\beta > 1). \tag{7.6.4}$$

Since

$$\sin m\pi u/\sin \pi u = e^{i(m-1)\pi u}+e^{i(m-3)\pi u}+\ldots+e^{-i(m-1)\pi u},$$

(7.6.3), with $\alpha = \beta$, gives

$$\int\limits_{0}^{\infty} \frac{\sin m\pi u}{\sin \pi u}\,\frac{du}{\Gamma(\alpha+u)\Gamma(\alpha-u)} = \frac{2^{2\alpha-3}}{\Gamma(2\alpha-1)} \quad (m\text{ odd}), \qquad 0 \quad (m\text{ even}). \tag{7.6.5}$$

The particular case $\alpha = n+1$ (n an integer) is

$$\int_0^\infty \frac{\sin m\pi u\, du}{u\left(1-\frac{u^2}{1^2}\right)\left(1-\frac{u^2}{2^2}\right)\cdots\left(1-\frac{u^2}{n^2}\right)} = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{2n!} \quad (m \text{ odd}), \qquad 0 \quad (m \text{ even}). \tag{7.6.6}$$

Again, apply (2.1.1) with

$$F(x) = \frac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)}, \qquad G(x) = \frac{1}{\Gamma(\gamma+x)\Gamma(\delta-x)}.$$

Then, by (7.6.3), if $\alpha+\beta+\gamma+\delta > 3$,

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}$$

$$= \frac{1}{2\pi}\frac{1}{\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_{-\pi}^{\pi} (2\cos\tfrac{1}{2}y)^{\alpha+\beta+\gamma+\delta-4} e^{\frac{1}{2}iy(\alpha-\beta-\gamma+\delta)}\, dy$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)\Gamma(\alpha+\delta-1)\Gamma(\beta+\gamma-1)}, \tag{7.6.7}$$

using (7.6.1) again. Here F and g are L^p if $2-\gamma-\delta < 1/p < \alpha+\beta-1$.

The formula (2.1.8) with the same functions and $x = \pi$, $\alpha+\delta = \beta+\gamma$, gives

$$\int_{-\infty}^{\infty} \frac{e^{-i\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\, dx$$

$$= \frac{1}{2\pi}\frac{e^{\frac{1}{2}i\pi(\alpha-\beta)}}{\Gamma(\alpha+\beta-1)\Gamma(\gamma+\delta-1)}\int_0^{\pi} (2\cos\tfrac{1}{2}y)^{\alpha+\beta-2}(2\sin\tfrac{1}{2}y)^{\gamma+\delta-2}\, dy$$

$$= \frac{e^{\frac{1}{2}i\pi(\alpha-\beta)}}{2\Gamma\{\frac{1}{2}(\alpha+\beta)\}\Gamma\{\frac{1}{2}(\gamma+\delta)\}\Gamma(\alpha+\delta-1)}. \tag{7.6.8}$$

In particular,

$$\int_0^\infty \frac{\cos\pi x}{\{\Gamma(\alpha+x)\Gamma(\alpha-x)\}^2}\, dx = \frac{1}{4\Gamma(2\alpha-1)\{\Gamma(\alpha)\}^2}. \tag{7.6.9}$$

Other integrals which may be evaluated in the same way are

$$\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\, dx$$

$$= \frac{2^{\alpha+\beta+\gamma+\delta-5} e^{\frac{1}{2}i\pi(\beta-\alpha)}\Gamma\{\frac{1}{2}(\alpha+\beta+\gamma+\delta-3)\}}{\sqrt{\pi}\,\Gamma\{\frac{1}{2}(\alpha+\beta)\}\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)} \tag{7.6.10}$$

provided that $2(\alpha-\beta) = \gamma-\delta$; if $\alpha+\beta+\gamma+\delta = 4$, then

$$\int\limits_{-\infty}^{\infty} \frac{\cos\pi(x+\beta+\gamma)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \frac{1}{2\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)\Gamma(2\beta+\gamma-2)}. \tag{7.6.11}$$

If $2(\alpha-\beta) = \gamma-\delta+k$, where k is ± 1 or ± 2, then

$$\int\limits_{-\infty}^{\infty} \frac{\sin\pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}\,dx = \pm\frac{2^{2\alpha-\gamma-3}}{\sqrt{\pi}\,\Gamma(\beta+\gamma-\alpha+\frac{1}{2})\Gamma(2\alpha+\delta-2)}. \tag{7.6.12}$$

If $3(\alpha-\beta) = \gamma-\delta+k$, where k is ± 1 or ± 2,

$$\int\limits_{-\infty}^{\infty} \frac{\sin\pi(2x+\alpha-\beta)}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+3x)\Gamma(\delta-3x)}\,dx = \pm\frac{3^{3\alpha+\delta-4}\Gamma(2\alpha-\beta+\delta-2)}{4\pi\Gamma(\gamma+\delta-1)\Gamma(3\alpha+\delta-3)}. \tag{7.6.13}$$

The sign on the right-hand side in each case is that of k.

We next take some integrals of a similar kind, but with Γ-functions in the numerator. Consider

$$I = \int\limits_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} e^{ixt}\,dx \quad (\mathbf{I}(\alpha) < 0)$$

$$= \pi \int\limits_{-\infty}^{\infty} \frac{1}{\Gamma(\beta+x)\Gamma(1-\alpha-x)} \frac{e^{ixt}}{\sin\pi(\alpha+x)}\,dx. \tag{7.6.14}$$

Now $$\frac{1}{\sin\pi(\alpha+x)} = \frac{2i}{e^{i\pi(\alpha+x)}-e^{-i\pi(\alpha+x)}} = 2i\sum_{m=0}^{\infty} e^{-i\pi(2m+1)(\alpha+x)}.$$

Hence $$I = 2i\pi \sum_{m=0}^{\infty} \int\limits_{-\infty}^{\infty} \frac{e^{ixt-i\pi(2m+1)(\alpha+x)}}{\Gamma(\beta+x)\Gamma(1-\alpha-x)}\,dx,$$

and these integrals are of the form (7.6.3), with $y = (2m+1)\pi-t$. Hence $I = 0$ if $t \leqslant 0$. If $t > 0$, the only non-zero term is that in which $m = [t/2\pi]$; the value of I may thus be obtained from (7.6.3).

We can now pass to

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x)e^{ixt}\,dx \tag{7.6.15}$$

in a similar way. This is

$$\pi \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(1-\beta+x)} \frac{e^{ixt}}{\sin \pi(\beta-x)}\,dx$$

$$= 2i\pi \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(1-\beta+x)} e^{ixt-i\pi(2m+1)(\beta-x)}\,dx$$

if $\mathbf{I}(\beta) < 0$. Hence (7.6.15) can be evaluated in terms of (7.6.14).

The above results may be used to evaluate some integrals involving Bessel functions, in which the order is the variable of integration.† Using (7.6.4), we have

$$\int_{-\infty}^{\infty} \frac{J_{\mu+x}(a)}{a^{\mu+x}} \frac{J_{\nu-x}(b)}{b^{\nu-x}}\,dx$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{m!\,n!} \left(\frac{1}{2}\right)^{\mu+\nu+2m+2n} \int_{-\infty}^{\infty} \frac{a^{2m}b^{2n}}{\Gamma(\mu+x+m+1)\Gamma(\nu-x+n+1)}\,dx$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{m!\,n!} \frac{a^{2m}b^{2n}}{2^{\mu+\nu+2m+2n}} \frac{2^{\mu+\nu+m+n}}{\Gamma(\mu+\nu+m+n+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r\Gamma(\mu+\nu+r+1)} \sum_{m=0}^{r} \frac{a^{2m}b^{2r-2m}}{m!\,(r-m)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r\Gamma(\mu+\nu+r+1)} \frac{(a^2+b^2)^r}{r!},$$

i.e.

$$\int_{-\infty}^{\infty} \frac{J_{\mu+x}(a)}{a^{\mu+x}} \frac{J_{\nu-x}(b)}{b^{\nu-x}}\,dx = \frac{J_{\mu+\nu}[\sqrt{\{2(a^2+b^2)\}}]}{\{\frac{1}{2}(a^2+b^2)\}^{\frac{1}{2}(\mu+\nu)}}. \tag{7.6.16}$$

In particular

$$\int_{-\infty}^{\infty} J_{\mu+x}(a)J_{\nu-x}(a)\,dx = J_{\mu+\nu}(2a). \tag{7.6.17}$$

The values of corresponding integrals containing a factor e^{iux} may be deduced in the same way from (7.6.3).

7.7. Mellin transforms. The simplest example of Mellin transforms is

$$f(x) = e^{-x}, \qquad \mathfrak{F}(s) = \Gamma(s) \quad (\sigma > 0). \tag{7.7.1}$$

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† Watson, § 13.8.

Here $f(x)x^{k-1}$ belongs to $L(0,\infty)$ if $k > 0$; and $\mathfrak{F}(s)$ belongs to $L(k-i\infty, k+i\infty)$ for $k > 0$.

Other straightforward examples are

$$1 \quad (x < a), \quad 0 \quad (x \geqslant a), \qquad a^s/s \qquad (\sigma > 0), \tag{7.7.2}$$

$$\log(a/x) \quad (x < a), \quad 0 \quad (x \geqslant a), \qquad a^s/s^2 \qquad (\sigma > 0), \tag{7.7.3}$$

$$\frac{1}{e^x-1}, \qquad \Gamma(s)\zeta(s) \qquad (\sigma > 1), \tag{7.7.4}$$

$$\frac{1}{e^x+e^{-x}}, \qquad \Gamma(s)L(s) \qquad (\sigma > 0), \tag{7.7.5}$$

where $L(s)$ is (9.12.1). Here $f(x)x^{k-1}$ belongs to $L(0,\infty)$ for $k > 0$ in each case except (7.7.4), when it is $k > 1$.

We also observe that if $f(x)$ and $\mathfrak{F}(s)$ are Mellin transforms so are $x^{\lambda}f(x)$ and $\mathfrak{F}(s+\lambda)$, and also $f(x^{\alpha})$ and $\frac{1}{\alpha}\mathfrak{F}\left(\frac{s}{\alpha}\right)$. This enables us to introduce parameters in each case.

Consider next the integral

$$f(z) = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}z^{-s}\,ds, \tag{7.7.6}$$

where $\mathbf{R}(a) > 0$, $\mathbf{R}(b) > 0$, c is not $0, -1, \dots$, and

$$0 < k < \min\{\mathbf{R}(a), \mathbf{R}(b)\}.$$

Since

$$\Gamma(s)\Gamma(a-s)\Gamma(b-s)/\Gamma(c-s) = O(e^{-\pi|t|}|t|^{\mathbf{R}(a+b-c)-1}), \qquad |z^{-s}| = r^{-\sigma}e^{\theta t},$$

the integral represents an analytic function of z, regular for $r > 0$, $-\pi < \theta < \pi$. If $z = x$, where $0 < x < 1$, it may be evaluated by moving the line of integration away to infinity on the left, and evaluating the residues at $s = 0, -1, \dots$. We obtain

$$\begin{aligned} f(x) &= 1-\frac{ab}{c\,1!}x+\frac{a(a+1)b(b+1)}{c(c+1)\,2!}x^2-\dots \\ &= F(a,b;\, c;\, -x) \end{aligned}$$

with the usual hypergeometric notation. For $x > 1$, $f(x)$ is therefore the analytic continuation of this function (and may, of course, be expressed as a sum of hypergeometric series by moving the contour the other way). We therefore obtain as Mellin transforms

$$f(x) = F(a,b;\, c;\, -x), \qquad \mathfrak{F}(s) = \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}$$

$$(0 < \sigma < \min\{\mathbf{R}(a), \mathbf{R}(b)\}). \tag{7.7.7}$$

Particular cases are

$$\frac{1}{1+x}, \qquad \frac{\pi}{\sin s\pi} \qquad (0<\sigma<1), \tag{7.7.8}$$

$$\frac{1}{(1+x)^a}, \qquad \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)} \qquad (0<\sigma<\mathbf{R}(a)), \tag{7.7.9}$$

$$\frac{1}{x}\log(1+x), \qquad \frac{\pi}{(1-s)\sin s\pi} \qquad (0<\sigma<1), \tag{7.7.10}$$

$$\frac{1}{(1+x)^m}P_{m-1}\left(\frac{1-x}{1+x}\right), \qquad \frac{\Gamma(s)\{\Gamma(m-s)\}^2}{\Gamma(1-s)\{\Gamma(m)\}^2} \qquad (0<\sigma<m), \tag{7.7.11}$$

where $P_n(x)$ is the Legendre polynomial of degree n.

In each case $\mathfrak{F}(\sigma+it)$ belongs to $L(-\infty,\infty)$ for the range of values of σ stated. In cases (7.7.8), (7.7.9), and (7.7.10) the integral

$$\int_0^\infty f(x)x^{s-1}\,dx \tag{7.7.12}$$

can easily be proved to be equal to $\mathfrak{F}(s)$.

Another Mellin pair of the same type is

$$f(x)=\frac{\{\sqrt{(x^2+1)}-x\}^a}{\sqrt{(x^2+1)}}, \qquad \mathfrak{F}(s)=2^{-s}\frac{\Gamma(s)\Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}s)}$$

$$(0<\sigma<\mathbf{R}(a)+1). \tag{7.7.13}$$

Here the integral (7.7.12) may be evaluated by putting

$$x=\tfrac{1}{2}y/\sqrt{(y+1)}.$$

Another class of Mellin transforms is

$$f(x)=\begin{cases}(1-x)^{a-1} & (0<x<1),\\ 0 & (x\geqslant 1),\end{cases} \qquad \mathfrak{F}(s)=\frac{\Gamma(s)\Gamma(a)}{\Gamma(s+a)} \qquad \left(\begin{matrix}\sigma>0,\\ \mathbf{R}(a)>0\end{matrix}\right), \tag{7.7.14}$$

$$\begin{cases}0 & (0<x\leqslant 1),\\ (x-1)^{-a} & (x>1),\end{cases} \qquad \frac{\Gamma(a-s)\Gamma(1-a)}{\Gamma(1-s)} \qquad (\sigma<\mathbf{R}(a)<1), \tag{7.7.15}$$

$$\begin{cases}0 \quad (0<x\leqslant 1),\\ \dfrac{\{x-\sqrt{(x^3-1)}\}^a+\{x-\sqrt{(x^2-1)}\}^{-a}}{\sqrt{(x^2-1)}}\\ (x>1),\end{cases} \qquad \frac{2^{-s}\Gamma(\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s-\frac{1}{2}a)}{\Gamma(1-s)} \qquad (\sigma<|\mathbf{R}(a)|+1), \tag{7.7.16}$$

$$\log\left|\frac{1+x}{1-x}\right|, \qquad \frac{\pi}{s}\tan\tfrac{1}{2}s\pi \qquad (-1<\sigma<1). \tag{7.7.17}$$

In each case $f(x)$ belongs to a different analytic function for $0 < x < 1$ and for $x > 1$, while $x^{k-1}f(x)$ belongs to $L(0,\infty)$ for some k. The evaluation of (7.7.12) is immediate in cases (7.7.14) and (7.7.15); for (7.7.16), put $x = \frac{1}{2}(y+1/y)$. For (7.7.17),

$$\int\limits_0^1 \log\left(\frac{1+x}{1-x}\right)x^{s-1}\,dx = 2\int\limits_0^1 \left(x+\frac{x^3}{3}+\dots\right)x^{s-1}\,dx = 2\left(\frac{1}{1+s}+\frac{1}{3(3+s)}+\dots\right),$$

$$\int\limits_1^\infty \log\left(\frac{x+1}{x-1}\right)x^{s-1}\,dx = \int\limits_0^1 \log\left(\frac{1+u}{1-u}\right)u^{1-s}\frac{du}{u^2} = 2\left(\frac{1}{1-s}+\frac{1}{3(3-s)}+\dots\right),$$

and $$\mathfrak{F}(s) = 4\left(\frac{1}{1^2-s^2}+\frac{1}{3^2-s^2}+\dots\right) = \frac{\pi}{s}\tan\tfrac{1}{2}s\pi.$$

7.8. Further gamma-function formulae. In (2.1.12) let $f(x) = x^a e^{-x}$, $\mathfrak{F}(s) = \Gamma(s+a)$, $g(x) = x^{b-1}e^{-x}$, $\mathfrak{G}(s) = \Gamma(s+b-1)$. Then

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(a+s)\Gamma(b-s)\,ds = \int\limits_0^\infty x^{a+b-1}e^{-2x}\,dx$$

$$= 2^{-a-b}\Gamma(a+b) \quad (-a < k < b). \tag{7.8.1}$$

This process and the following ones are justified by Theorem 42. The result is a particular case of the reciprocity (7.7.9).

Taking $b = a$ and the line of integration the imaginary axis, we obtain

$$\int\limits_0^\infty |\Gamma(a+it)|^2\,dt = 2^{-2a}\pi\Gamma(2a) \quad (a > 0), \tag{7.8.2}$$

and there are similar particular cases of the following formulae.

Next let

$$f(x) = \frac{x^b}{(1+x)^a}, \qquad \mathfrak{F}(s) = \frac{\Gamma(b+s)\Gamma(a-b-s)}{\Gamma(a)},$$

and so g, $\mathfrak{G}$, with c, d for a, b. Then

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(b+s)\Gamma(a-b-s)\Gamma(d+1-s)\Gamma(c-d-1+s)}{\Gamma(a)\Gamma(c)}\,ds$$

$$= \int\limits_0^\infty \frac{x^{b+d}}{(1+x)^{a+c}}\,dx = \frac{\Gamma(b+d+1)\Gamma(a+c-b-d-1)}{\Gamma(a+c)},$$

or, writing $c-d-1 = \alpha$, $b = \beta$, $1+d = \gamma$, $a-b = \delta$,

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds$$

$$= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \quad (-\alpha < k, -\beta < k, \gamma > k, \delta > k). \tag{7.8.3†}$$

Let

$$f(x) = x^{b}(1-x)^{a-1} \ (0 < x < 1), \quad 0 \ (x > 1), \qquad \mathfrak{F}(s) = \frac{\Gamma(b+s)\Gamma(a)}{\Gamma(a+b+s)},$$

and so g, $\mathfrak{G}$, with c, d for a, b. Then

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(b+s)\Gamma(a)\Gamma(1+d-s)\Gamma(c)}{\Gamma(a+b+s)\Gamma(c+d+1-s)}\,ds = \int\limits_{0}^{1} x^{b+d}(1-x)^{a+c-2}\,dx$$

$$= \frac{\Gamma(b+d+1)\Gamma(a+c-1)}{\Gamma(a+b+c+d)},$$

or, writing $a = \beta-\alpha$, $b = \alpha$, $c = \delta-\gamma$, $d = \gamma-1$,

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\alpha+s)\Gamma(\gamma-s)}{\Gamma(\beta+s)\Gamma(\delta-s)}\,ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\delta-\alpha-\gamma-1)}{\Gamma(\beta-\alpha)\Gamma(\delta-\gamma)\Gamma(\beta+\delta-1)}$$

$$(-\alpha < k, -\beta < k, \gamma > k, \delta > k). \tag{7.8.4}$$

Defining $f(x)$ as in the last example and $g(x)$ as in (7.8.3), we obtain

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(b+s)\Gamma(a)\Gamma(d+1-s)\Gamma(c-d-1+s)}{\Gamma(a+b+s)\Gamma(c)}\,ds$$

$$= \int\limits_{0}^{1} \frac{x^{b+d}(1-x)^{a-1}}{(1+x)^{c}}\,dx.$$

The integral can be evaluated in finite terms if $c = 1-a$. It is then

$$\int\limits_{0}^{1} x^{b+d}(1-x^{2})^{a-1}\,dx = \tfrac{1}{2}\int\limits_{0}^{1} y^{\frac{1}{2}(b+d-1)}(1-y)^{a-1}\,dy = \frac{1}{2}\,\frac{\Gamma\{\frac{1}{2}(b+d+1)\}\Gamma(a)}{\Gamma\{a+\frac{1}{2}(b+d+1)\}}.$$

Putting $b = \alpha$, $a = 1-\beta-\gamma$, $d = \gamma-1$, we obtain

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)}{\Gamma(1+\alpha-\beta-\gamma+s)}\,ds = \frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\gamma)\Gamma(\beta+\gamma)}{2\Gamma(1-\beta+\frac{1}{2}\alpha-\frac{1}{2}\gamma)}$$

$$(-\alpha < k, -\beta < k, \gamma > k, \beta+\gamma-\alpha-1 < k). \tag{7.8.5}$$

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† Barnes (1); see Whittaker and Watson, *Modern Analysis*, § 14.52.

Another formula of Barnes† is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma+s)\Gamma(\delta-s)\Gamma(-s)}{\Gamma(\epsilon+s)}\,ds$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\delta)\Gamma(\gamma+\delta)}{\Gamma(\epsilon-\alpha)\Gamma(\epsilon-\beta)\Gamma(\epsilon-\gamma)}, \tag{7.8.6}$$

where $\alpha+\beta+\gamma+\delta = \epsilon.$

To prove this, we use the formula

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\mathfrak{F}_1(s)\mathfrak{F}_2(s)\mathfrak{F}_3(s)\,ds = \int\limits_0^\infty\int\limits_0^\infty f_1(u)f_2(v)f_3\!\left(\frac{1}{uv}\right)\frac{dudv}{uv},$$

derived from (2.1.18) with $n = 2$. Take

$$f_1(x) = \frac{x^\alpha}{(1+x)^{\alpha+\delta}}, \qquad \mathfrak{F}_1(s) = \frac{\Gamma(\alpha+s)\Gamma(\delta-s)}{\Gamma(\alpha+\delta)},$$

$$f_2(x) = \frac{x^\beta}{(1+x)^\beta}, \qquad \mathfrak{F}_2(s) = \frac{\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\beta)},$$

$$f_3(x) = \begin{cases} x^\gamma(1-x)^{\epsilon-\gamma-1} & (0<x<1) \\ 0 & (x>1), \end{cases} \qquad \mathfrak{F}_3(s) = \frac{\Gamma(\gamma+s)\Gamma(\epsilon-\gamma)}{\Gamma(\epsilon+s)}.$$

Denoting the left-hand side of (7.8.6) by I, we obtain

$$\frac{\Gamma(\epsilon-\gamma)}{\Gamma(\alpha+\delta)\Gamma(\beta)}I = \iint\limits_{uv>1}\frac{u^\alpha}{(1+u)^{\alpha+\delta}}\frac{v^\beta}{(1+v)^\beta}\frac{1}{(uv)^\gamma}\left(1-\frac{1}{uv}\right)^{\epsilon-\gamma-1}\frac{dudv}{uv}.$$

Putting $u = \dfrac{1}{x}-1$, $v = \dfrac{1}{y}-1$, the right-hand side becomes

$$\iint\limits_{x+y<1} x^{\gamma+\delta-1}y^{\gamma-1}(1-x)^{\alpha-\epsilon}(1-y)^{\beta-\epsilon}(1-x-y)^{\epsilon-\gamma-1}\,dxdy.$$

Putting $y = z(1-x)$, we obtain

$$\int\limits_0^1 z^{\gamma-1}(1-z)^{\epsilon-\gamma-1}\,dz\int\limits_0^1 x^{\gamma+\delta-1}(1-x)^{\alpha-1}(1-z+zx)^{\beta-\epsilon}\,dx.$$

The inner integral can be evaluated in terms of Γ-functions if $\alpha+\beta+\gamma+\delta = \epsilon$. It is then equal to‡

$$\frac{\Gamma(\alpha)\Gamma(\gamma+\delta)}{\Gamma(\alpha+\gamma+\delta)}\frac{1}{(1-z)^\alpha}.$$

Hence we obtain

$$\frac{\Gamma(\alpha)\Gamma(\gamma+\delta)}{\Gamma(\alpha+\gamma+\delta)}\int\limits_0^1 z^{\gamma-1}(1-z)^{\epsilon-\gamma-\alpha-1}\,dz = \frac{\Gamma(\alpha)\Gamma(\gamma+\delta)}{\Gamma(\alpha+\gamma+\delta)}\frac{\Gamma(\gamma)\Gamma(\epsilon-\gamma-\alpha)}{\Gamma(\epsilon-\alpha)},$$

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† Barnes (2).

‡ See Titchmarsh, *Theory of Functions*, Chap. I, Ex. 19.

and the result follows. The necessary inversions are all justified by absolute convergence.

7.9. Bessel functions. In (7.4.1) we may take $a = s$ to be complex, provided that $0 < \sigma < \nu+\frac{3}{2}$. We thus obtain the Mellin transforms

$$x^{-\nu}J_\nu(x), \qquad \frac{2^{s-\nu-1}\Gamma(\frac{1}{2}s)}{\Gamma(\nu-\frac{1}{2}s+1)} \qquad (0 < \sigma < \nu+\tfrac{3}{2}). \qquad (7.9.1)†$$

Equivalent pairs are

$$J_\nu(x), \qquad \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+1)} \qquad (-\nu < \sigma < \tfrac{3}{2}), \qquad (7.9.2)$$

$$x^{\nu}J_\nu(x), \qquad \frac{2^{s+\nu-1}\Gamma(\frac{1}{2}s+\nu)}{\Gamma(1-\frac{1}{2}s)} \qquad (-2\nu < \sigma < \tfrac{3}{2}-\nu), \qquad (7.9.3)$$

and

$$x^{\frac{1}{2}}J_\nu(x), \qquad \frac{2^{s-\frac{1}{2}}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} \qquad (-\nu-\tfrac{1}{2} < \sigma < 1). \qquad (7.9.4)$$

Taking $\nu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ in the last pair, we obtain

$$\cos x, \qquad 2^{s-1}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)} = \Gamma(s)\cos\tfrac{1}{2}s\pi \quad (0 < \sigma < 1), \qquad (7.9.5)$$

$$\sin x, \qquad 2^{s-1}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)} = \Gamma(s)\sin\tfrac{1}{2}s\pi \quad (-1 < \sigma < 1). \qquad (7.9.6)$$

We define
$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}.$$

By (7.9.2) the Mellin transform of $Y_\nu(x)$ is

$$\frac{1}{\sin\nu\pi}\left\{\frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+1)}\cos\nu\pi - \frac{2^{s-1}\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\nu-\frac{1}{2}s)}\right\}$$
$$= \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)}{\pi\sin\nu\pi}\{\sin(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\pi\cos\nu\pi - \sin(\tfrac{1}{2}s+\tfrac{1}{2}\nu)\pi\}$$
$$= -2^{s-1}\pi^{-1}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\cos(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\pi.$$

Hence we have the Mellin pairs

$$Y_\nu(x), \qquad -2^{s-1}\pi^{-1}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\cos(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\pi \quad (|\nu| < \sigma < \tfrac{3}{2}), \qquad (7.9.7)‡$$

$$x^{-\nu}Y_\nu(x), \qquad -2^{s-\nu-1}\pi^{-1}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s-\nu)\cos(\tfrac{1}{2}s-\nu)\pi \quad (\nu+|\nu| < \sigma < \nu+\tfrac{3}{2}). \qquad (7.9.8)$$

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† Watson, § 6.5 (7).

‡ Ibid., § 13.24 (5).

From (7.9.2) we also deduce

$$J_\nu(x)+J_{-\nu}(x), \qquad 2^s\pi^{-1}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\sin\tfrac{1}{2}s\pi\cos\tfrac{1}{2}\nu\pi \quad (|\nu|<\sigma<\tfrac{3}{2}), \tag{7.9.9}$$

$$J_\nu(x)-J_{-\nu}(x), \qquad -2^s\pi^{-1}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\cos\tfrac{1}{2}s\pi\sin\tfrac{1}{2}\nu\pi \quad (|\nu|<\sigma<\tfrac{3}{2}). \tag{7.9.10}$$

Again, by (7.9.1) and (7.9.8),

$$x^{-\nu}\{J_\nu(x)+iY_\nu(x)\}$$

$$=\frac{1}{2\pi^2 i}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-\nu-1}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s-\nu)\{\sin(\tfrac{1}{2}s-\nu)\pi-i\cos(\tfrac{1}{2}s-\nu)\pi\}x^{-s}\,ds$$

$$=-\frac{1}{2\pi^2}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-\nu-1}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s-\nu)e^{i(\frac{1}{2}s-\nu)\pi}x^{-s}\,ds,$$

and here we can (by analytic continuation) replace x by ix. We obtain†

$$x^{-\nu}e^{-\frac{1}{2}i\pi\nu}\frac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(x)$$

$$=-\frac{1}{2\pi^2}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-\nu-1}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s-\nu)e^{i\pi(\frac{1}{2}s-\nu)}x^{-s}e^{-\frac{1}{2}i\pi s}\,ds,$$

so that we obtain as Mellin transforms

$$x^{-\nu}K_\nu(x), \qquad 2^{s-\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s-\nu) \quad (\sigma>\max(0,2\nu)). \tag{7.9.11‡}$$

An equivalent pair is

$$x^{\nu}K_\nu(x), \qquad 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu) \quad (\sigma>\max(0,-2\nu)). \tag{7.9.12}$$

Hence we verify that $K_\nu(x)$ is an even function of ν. For $\nu=\frac{1}{2}$ (7.9.12) reduces to (7.7.1).

From (7.4.2) we obtain the pair

$$x^{-\nu}\mathbf{H}_\nu(x), \qquad \frac{2^{s-\nu-1}\Gamma(\frac{1}{2}s)\tan\frac{1}{2}s\pi}{\Gamma(\nu-\frac{1}{2}s+1)} \quad (-1<\sigma<\nu+\tfrac{3}{2}), \tag{7.9.13}$$

and variants of this can easily be obtained in the above way.

To justify the inversion formulae in the above cases, consider e.g. (7.9.1). We obtained (7.4.1) directly; the inverse formula is

$$\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int\limits_{k-i\lambda}^{k+i\lambda}\frac{2^{s-\nu-1}\Gamma(\frac{1}{2}s)}{\Gamma(\nu-\frac{1}{2}s+1)}x^{-s}\,ds \doteq \frac{J_\nu(x)}{x^\nu}. \tag{7.9.14}$$

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† Watson, § 3.7 (8).

‡ Ibid., § 6.5 (3–6).

This follows from Theorem 28 if $0 < k < \nu+\frac{1}{2}$, and from Theorem 30 if $0 < k \leqslant \nu+1$. The leading terms in the asymptotic expansion of $x^{-\nu}J_\nu(x)$ are of the form $x^{-\nu-\frac{1}{2}}(a\cos x+b\sin x)$. Hence, in Theorem 30,

$$\phi(x) = e^{(k-\nu-\frac{1}{2})x}, \qquad \psi(x) = e^x,$$

and the crucial condition (1.12.1) is satisfied if $k \leqslant \nu+1$.

We might begin by proving (7.9.14) by the calculus of residues. We have then to deduce (7.4.1). We have as $t \to \infty$

$$\frac{2^{s-\nu-1}\Gamma(\frac{1}{2}s)}{\Gamma(\nu-\frac{1}{2}s+1)} = e^{\frac{1}{2}(\nu-1)i\pi+i(t\log t-t)}t^{\sigma-\nu-1}\left\{1+\frac{a}{t}+O\left(\frac{1}{t^2}\right)\right\}.$$

The result follows from Theorem 29 if $0 < k < \nu$. We can also apply Theorem 11; here

$$\phi(t) = t^{k-\nu-1}, \qquad \psi(t) = t\log t-t,$$

and (1.12.1) is satisfied if $k < \nu+\frac{3}{2}$.

7.10. Products of Bessel functions. By (2.1.16) and (7.9.1) the Mellin transform of $x^{-\mu-\nu}J_\mu(x)J_\nu(x)$ is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{w-\mu-1}\frac{\Gamma(\frac{1}{2}w)}{\Gamma(1+\mu-\frac{1}{2}w)}2^{s-w-\nu-1}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2}w)}{\Gamma(1+\nu-\frac{1}{2}s+\frac{1}{2}w)}\,dw,$$

and putting $w = 2w'$ and using (7.8.4), we obtain the Mellin pair†

$$\frac{J_\mu(x)J_\nu(x)}{x^{\mu+\nu}}, \qquad \frac{2^{s-\mu-\nu-1}\Gamma(\frac{1}{2}s)\Gamma(1+\mu+\nu-s)}{\Gamma(1+\nu-\frac{1}{2}s)\Gamma(1+\mu-\frac{1}{2}s)\Gamma(1+\mu+\nu-\frac{1}{2}s)}. \tag{7.10.1}$$

Similarly, by (7.9.11), the Mellin transform of $x^{-\mu-\nu}K_\mu(x)K_\nu(x)$ is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{w-\mu-2}\Gamma(\tfrac{1}{2}w)\Gamma(\tfrac{1}{2}w-\mu)2^{s-w-\nu-2}\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}w)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}w-\nu)\,dw,$$

and, using (7.8.3), we obtain the Mellin pair

$$\frac{K_\mu(x)K_\nu(x)}{x^{\mu+\nu}}, \qquad \frac{2^{s-\mu-\nu-3}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s-\mu)\Gamma(\frac{1}{2}s-\nu)\Gamma(\frac{1}{2}s-\mu-\nu)}{\Gamma(s-\mu-\nu)}. \tag{7.10.2}$$

From (7.9.1) and (7.9.11), the Mellin transform of $x^{-2\nu}J_\nu(x)K_\nu(x)$ is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{w-\nu-2}\Gamma(\tfrac{1}{2}w)\Gamma(\tfrac{1}{2}w-\nu)2^{s-w-\nu-1}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2}w)}{\Gamma(1+\nu-\frac{1}{2}s+\frac{1}{2}w)}\,dw,$$

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† Watson 13.41 (1), (2), 13.33 (1).

and, using (7.8.5), we obtain the Mellin pair

$$x^{-2\nu}J_\nu(x)K_\nu(x), \qquad \frac{2^{s-2\nu-2}\Gamma(\frac{1}{4}s)\Gamma(\frac{1}{2}s-\nu)}{\Gamma(1+\nu-\frac{1}{4}s)}. \tag{7.10.3}$$

By combining particular cases of (7.10.1), we obtain as Mellin transforms

$$J_\nu(x)Y_\nu(x), \qquad -\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\nu)}{\Gamma(\frac{1}{2}s+\frac{1}{2})\Gamma(1+\nu-\frac{1}{2}s)}. \tag{7.10.4}$$

Other particular cases of (7.10.1) give the Mellin transforms

$$\cos x\, J_\nu(x), \qquad \frac{2^{s-1}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-s)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}, \tag{7.10.5}$$

$$\sin x\, J_\nu(x), \qquad \frac{2^{s-1}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{2})\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(1-\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}s)}. \tag{7.10.6}$$

Combining these, we have

$$e^{ix}J_\nu(x) = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{2^{-s}}{\sqrt{\pi}}\frac{\Gamma(s+\nu)\Gamma(\frac{1}{2}-s)}{\Gamma(1+\nu-s)}e^{\frac{1}{2}i\pi(s+\nu)}x^{-s}\,ds.$$

As in (7.9.11), we may now replace x by ix, and obtain the Mellin transforms

$$e^{-x}I_\nu(x), \qquad \frac{\Gamma(s+\nu)\Gamma(\frac{1}{2}-s)}{2^s\pi^{\frac{1}{2}}\Gamma(1+\nu-s)}. \tag{7.10.7}$$

Again, from (7.10.1) and (7.10.4), the Mellin transform of

$$J_\nu(x)\{J_\nu(x)+iY_\nu(x)\}$$

is

$$-\frac{ie^{\frac{1}{2}is\pi}}{2\pi^{\frac{1}{2}}}\frac{\Gamma(\frac{1}{2}s+\nu)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(1+\nu-\frac{1}{2}s)},$$

and hence, replacing x by ix, we obtain as Mellin transforms

$$I_\nu(x)K_\nu(x), \qquad \frac{\Gamma(\frac{1}{2}s+\nu)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{4\sqrt{\pi}\,\Gamma(1+\nu-\frac{1}{2}s)}. \tag{7.10.8}$$

Similarly, the Mellin transform of $e^{-ix}\{J_\nu(x)+iY_\nu(x)\}$ is

$$-\frac{2^{1-s}i}{\pi^{\frac{1}{2}}}e^{\frac{1}{2}i\pi(s-\nu)}\cos\nu\pi\,\Gamma(\tfrac{1}{2}-s)\Gamma(s+\nu)\Gamma(s-\nu),$$

and replacing x by ix we obtain

$$e^{x}K_\nu(x), \qquad 2^{-s}\pi^{-\frac{1}{2}}\cos\nu\pi\Gamma(\tfrac{1}{2}-s)\Gamma(s+\nu)\Gamma(s-\nu). \tag*{(7.10.9)†}$$

The processes of this section can be justified by Theorem 73; for example, $x^k x^{-\mu} J_\mu(x)$ belongs to $\mathfrak{L}^2$ if $0 < k < \mu+\frac{1}{2}$, and $x^{\sigma-k}x^{-\nu}J_\nu(x)$ belongs to $\mathfrak{L}^2$ if $0 < \sigma-k < \nu+\frac{1}{2}$. We can choose σ and k to satisfy these conditions if $\mu > -\frac{1}{2}$, $\nu > -\frac{1}{2}$. This would not include (7.10.5) directly; for this we could consider first $(1-\cos x)J_\nu(x)$, $x^k(1-\cos x)$ belonging to $\mathfrak{L}^2$ if $-2 < k < 0$. We can of course also extend the ranges of validity of the formulae by analytic continuation.

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† Ibid., § 6.51.

7.11. Integrals involving Bessel functions. We are now in a position to evaluate a large class of integrals involving Bessel functions.

By transformation from (7.7.15), we have the Mellin pair

$$(x^2-1)^{-\nu-\frac{1}{2}} \quad (x > 1), \quad 0 \quad (x < 1), \qquad \frac{\Gamma(\nu+\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}-\nu)}{2\Gamma(1-\frac{1}{2}s)}. \tag{7.11.1}$$

Using this and (7.9.6), (2.1.23) gives

$$\int\limits_1^\infty \frac{\sin ax}{(x^2-1)^{\nu+\frac{1}{2}}}\,dx = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-1}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\,\frac{\Gamma(\nu+\frac{1}{2}s)\Gamma(\frac{1}{2}-\nu)}{2\Gamma(\frac{1}{2}+\frac{1}{2}s)}\,a^{-s}\,ds$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-\nu)}{4\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-1}\frac{\Gamma(\nu+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\,a^{-s}\,ds$$

$$= 2^{-\nu-1}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\nu)a^\nu J_\nu(a) \quad (-\tfrac{1}{2} < \nu < \tfrac{1}{2}) \tag{7.11.2}$$†

by (7.9.3). The formula is the sine-reciprocal of (7.4.3). Similarly, using (7.9.5) and (7.9.7),

$$\int\limits_1^\infty \frac{\cos ax}{(x^2-1)^{\nu+\frac{1}{2}}}\,dx = -2^{-\nu-1}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\nu)a^\nu Y_\nu(a) \quad (-\tfrac{1}{2} < \nu < \tfrac{1}{2}). \tag{7.11.3}$$‡

These processes come under Theorem 42 if $\frac{1}{4} < \nu < \frac{1}{2}$; for

$$\mathfrak{F}(s) = 2^{s-1}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}a^{-s} = O(|t|^{k-\frac{1}{2}}),$$

which is L if $k < -\frac{1}{2}$; and $x^{-k}(x^2-1)^{-\nu-\frac{1}{2}}$ is L if $-\frac{1}{2}k < \nu < \frac{1}{2}$. The result can be extended to the full range $-\frac{1}{2} < \nu < \frac{1}{2}$ by analytic continuation, or we can use Theorem 43. For $0 < \sigma < 1$

$$\int\limits_a^1 \sin x\, x^{s-1}\,dx = O(1)$$

and $$\int\limits_1^b \sin x\, x^{s-1}\,dx = \Big[-\cos x\, x^{s-1}\Big]_1^b + (s-1)\int\limits_1^b \cos x\, x^{s-2}\,dx$$

$$= O(1)+O(|t|) = O(|t|).$$

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† Watson, 6.13 (3).

‡ Ibid., 6.13 (4).

Also, for $\sigma = 1-k$,

$$\int\limits_1^b \frac{x^{s-1}}{(x^2-1)^{\nu+\frac{1}{2}}} = O(1)$$

if $-\tfrac{1}{2}k < \nu < \tfrac{1}{2}$. The other conditions are plainly fulfilled if $-\tfrac{1}{2} < \nu < \tfrac{1}{2}$. By taking k arbitrarily near to 1 the result follows.

Again, from the Mellin pairs

$$x^{\nu+1}J_\nu(ax), \qquad 2^{s+\nu}a^{-s-\nu-1}\frac{\Gamma(\frac{1}{2}s+\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}, \tag{7.11.4}$$

$$(x^2+1)^{-\mu-1}, \qquad \frac{\Gamma(\frac{1}{2}s)\Gamma(\mu+1-\frac{1}{2}s)}{2\Gamma(\mu+1)} \tag{7.11.5}$$

(transformations of (7.9.3) and (7.7.9)), we obtain

$$\begin{aligned}
&\int\limits_0^\infty \frac{x^{\nu+1}J_\nu(ax)}{(x^2+1)^{\mu+1}}\,dx \\
&= \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{s+\nu}a^{-s-\nu-1}\frac{\Gamma(\frac{1}{2}s+\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\mu+\frac{1}{2}+\frac{1}{2}s)}{2\Gamma(\mu+1)}\,ds \\
&= \frac{1}{4\pi i\Gamma(\mu+1)}\int\limits_{k+2\nu+1-i\infty}^{k+2\nu+1+i\infty} 2^{s'-\nu-1}a^{-s'+\nu}\Gamma(\tfrac{1}{2}s')\Gamma(\tfrac{1}{2}s'+\mu-\nu)\,ds' \\
&= \frac{a^\mu K_{\mu-\nu}(a)}{2^\mu\Gamma(\mu+1)}
\end{aligned} \tag{7.11.6}$$ †

by (7.9.12). Here the $\mathfrak{F}(s)$ of (7.11.4) is $L(k-i\infty,\ k+i\infty)$ if $-2\nu-1 < k < -\nu-1$; and $x^{-k}(x^2+1)^{-\mu-1}$ is L if $-2\mu-1 < k < 1$. These conditions are consistent if $0 < \nu < 2\mu$, and the result then follows from Theorem 42. The formula is actually valid if $-1 < \nu < 2\mu+\tfrac{3}{2}$. It can be extended to the full range either by analytic continuation or by Theorem 43.

The following examples can be obtained in a similar way, and present no particular difficulty: ‡

$$\int\limits_1^\infty e^{-ax}(x^2-1)^{\nu-\frac{1}{2}}\,dx = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\frac{K_\nu(a)}{a^\nu}, \tag{7.11.7}$$

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† Ibid., 13.6 (2).

‡ The corresponding formulae in Watson are 6.15 (4), 13.2 (5), 13.2 (8), 13.3 (4), 13.3 (5), 13.45 (2), 13.6 (3), 13.6 (5).

$$\int_0^\infty e^{-ax}J_\nu(x)x^\nu\,dx = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(a^2+1)^{\nu+\frac{1}{2}}}, \tag{7.11.8}$$

(= 7.4.8)

$$\int_0^\infty e^{-ax}J_\nu(x)\,dx = \frac{\{\sqrt{(1+a^2)}-a\}^\nu}{\sqrt{(1+a^2)}}, \tag{7.11.9}$$

$$\int_0^\infty J_\nu(ax)e^{-bx^2}x^{\nu+1}\,dx = \frac{a^\nu}{(2b)^{\nu+1}}e^{-\frac{1}{4}a^2/b}, \tag{7.11.10}$$

$$\int_0^\infty J_{2\nu}(ax)e^{-bx^2}\,dx = \frac{\pi^{\frac{1}{2}}}{2b^{\frac{1}{2}}}e^{-a^2/8b}I_\nu\left(\frac{a^2}{8b}\right), \tag{7.11.11}$$

$$\int_0^\infty K_\mu(at)J_\nu(bt)t^{\mu+\nu+1}\,dt = \frac{(2a)^\mu(2b)^\nu\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}, \tag{7.11.12}$$

$$\int_0^\infty \frac{J_\nu(ax)}{\sqrt{(x^2+1)}}\,dx = I_{\frac{1}{2}\nu}(\tfrac{1}{2}a)K_{\frac{1}{2}\nu}(\tfrac{1}{2}a), \tag{7.11.13}$$

$$\int_0^\infty \frac{x^{\nu+1}J_\nu(ax)}{(x^4+4)^{\nu+\frac{1}{2}}}\,dx = \frac{(\frac{1}{2}a)^\nu\pi^{\frac{1}{2}}}{2^{2\nu}\Gamma(\nu+\frac{1}{2})}J_\nu(a)K_\nu(a). \tag{7.11.14}$$

A more delicate example is

$$\begin{aligned}\int_0^\infty J_\nu(ax)J_{\nu+1}(x)\,dx &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+1)}\,\frac{2^{-s}\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+1)}a^{-s}\,ds\\ &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{a^{-s}}{s+\nu}\,ds \quad (k>-\nu)\\ &= a^\nu\ (0<a<1),\quad \tfrac{1}{2}\ (a=1),\quad 0\ (a>1).\end{aligned} \tag{7.11.15†}$$

If $a \neq 1$ this may be justified by Theorem 43, with

$$\chi(\xi) = \int_0^\infty J_\nu(ax)J_{\nu+1}(\xi x)\,dx.$$

The conditions may be verified as in the proof of (7.11.1). If $a = 1$, $\chi(\xi)$ is discontinuous at $\xi = 1$, and the method fails to evaluate $\chi(1)$. We can fill in this case by proving directly that $\chi(1) = \frac{1}{2}\{\chi(1+0)+\chi(1-0)\}$; or we can apply the method in the opposite direction, which gives the whole result, but with more tedious details; or we can write

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† Watson, § 13.42 (8).

$$\int_0^\infty J_\nu(x)J_{\nu+1}(x)\,dx = \int_0^\infty \frac{d}{dx}\{x^{\nu+1}J_{\nu+1}(x)\}.x^{\nu+1}J_{\nu+1}(x)\,\frac{dx}{x^{2\nu+2}}$$

$$= -\int_0^\infty x^{\nu+1}J_{\nu+1}(x)\left\{\frac{x^{\nu+1}J_\nu(x)}{x^{2\nu+2}}-(2\nu+2)\frac{x^{\nu+1}J_{\nu+1}(x)}{x^{2\nu+3}}\right\}dx,$$

so that $$\int_0^\infty J_\nu(x)J_{\nu+1}(x)\,dx = (\nu+1)\int_0^\infty \frac{J_{\nu+1}^2(x)}{x}\,dx.$$

The last integral is $1/(2\nu+2)$, by taking $\nu = \mu$, $s = 2\mu$, in (7.10.1).

As an example on (7.8.6), we have, by (7.9.12) and (7.10.8),

$$\int_0^\infty x^{-\frac{1}{2}}K_\mu(2x)I_\nu(x)K_\nu(x)\,dx$$

$$= \frac{1}{8\sqrt{\pi}}\,\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\mu-\tfrac{1}{4})\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu-\tfrac{1}{4})\times$$

$$\times\frac{\Gamma(\frac{1}{2}+\nu-\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\nu+\frac{1}{2}s)}\,ds$$

$$= \frac{1}{4\sqrt{\pi}}\,\frac{1}{2\pi i}\int_{\frac{1}{2}(k-1)-i\infty}^{\frac{1}{2}(k-1)+i\infty} \Gamma(s'-\tfrac{1}{2}\mu+\tfrac{1}{4})\Gamma(s'+\tfrac{1}{2}\mu+\tfrac{1}{4})\times$$

$$\times\frac{\Gamma(\nu-s')\Gamma(\frac{1}{2}+s')\Gamma(-s')}{\Gamma(1+\nu+s')}\,ds'$$

$$= \frac{\Gamma(\frac{1}{4}-\frac{1}{2}\mu)\Gamma(\frac{1}{4}+\frac{1}{2}\mu)\Gamma(\nu+\frac{1}{4}-\frac{1}{2}\mu)\Gamma(\nu+\frac{1}{4}+\frac{1}{2}\mu)}{4\Gamma(\frac{3}{4}+\nu+\frac{1}{2}\mu)\Gamma(\frac{3}{4}+\nu-\frac{1}{2}\mu)}. \tag{7.11.16}$$

From (7.7.11), and (7.11.4) with $\nu = 0$, we obtain

$$\int_0^\infty \frac{1}{(1+x^2)^n}P_{n-1}\left(\frac{1-x^2}{1+x^2}\right)xJ_0(ax)\,dx$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^s\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)\{\Gamma(n-\frac{1}{2}+\frac{1}{2}s)\}^2}{2\Gamma(\frac{1}{2}+\frac{1}{2}s)\{\Gamma(n)\}^2}\,a^{-s-1}\,ds$$

$$= \frac{1}{2\pi i\{\Gamma(n)\}^2}\int_{k-i\infty}^{k+i\infty} 2^{s-1}\{\Gamma(n-\tfrac{1}{2}+\tfrac{1}{2}s)\}^2a^{-s-1}\,ds$$

$$= \frac{1}{2\pi i\{\Gamma(n)\}^2}\int\limits_{k+2n-1-i\infty}^{k+2n-1+i\infty} 2^{s'-2n}\{\Gamma(\tfrac{1}{2}s')\}^2 a^{-s'-2+2n}\,ds'$$

$$= \frac{(\frac{1}{2}a)^{2n-2}}{\{\Gamma(n)\}^2}K_0(a). \tag{7.11.17}$$

7.12. Some non-absolutely convergent integrals. We have, by (7.9.5) and (7.9.6), the Mellin transforms

$$\cos x^{\alpha}, \qquad \frac{1}{\alpha}\Gamma\left(\frac{s}{\alpha}\right)\cos\frac{s\pi}{2\alpha} \tag{7.12.1}$$

$$\sin x^{\alpha}, \qquad \frac{1}{\alpha}\Gamma\left(\frac{s}{\alpha}\right)\sin\frac{s\pi}{2\alpha}. \tag{7.12.2}$$

From (7.12.1), with $\alpha = 1$ and $\alpha = 2$, we obtain

$$\int\limits_0^{\to\infty} \cos x^2 \cos ax\,dx = \frac{1}{4\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(s)\cos\tfrac{1}{2}s\pi\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\cos\tfrac{1}{4}\pi(1-s)a^{-s}\,ds$$

$$= \frac{1}{4\pi i}\int\limits_{k-i\infty}^{k+i\infty} 2^{s-1}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s)\cos\tfrac{1}{4}\pi(1-s)a^{-s}\,ds$$

$$= \tfrac{1}{2}\pi^{\frac{1}{2}}\cos\tfrac{1}{4}(\pi-a^2), \tag{7.12.3}$$

by (7.12.1) and (7.12.2) with $\alpha = 2$. Similarly,

$$\int\limits_0^{\to\infty} \sin x^2 \cos ax\,dx = \tfrac{1}{2}\pi^{\frac{1}{2}}\sin\tfrac{1}{4}(\pi-a^2). \tag{7.12.4}$$

The results are equivalent to (7.1.8) and (7.1.9). The process is justified by Theorem 39. As in § 7.11 we have

$$\int\limits_{\lambda}^{\mu} x^{s-1}\cos x\,dx = O(|t|)$$

for all λ and μ. Also

$$\int\limits_{\lambda}^{\mu} \cos(ax-x^2)\,dx = \int\limits_{\lambda}^{\frac{1}{2}a-1} + \int\limits_{\frac{1}{2}a-1}^{\frac{1}{2}a+1} + \int\limits_{\frac{1}{2}a+1}^{\mu} = J_1+J_2+J_3$$

(with obvious modifications if $\lambda > \frac{1}{2}a-1$ or $\mu < \frac{1}{2}a+1$). Plainly $J_2 = O(1)$; and

$$J_3 = \int\limits_{x=\frac{1}{2}a+1}^{\mu} \frac{d\sin(ax-x^2)}{a-2x} = -\frac{1}{2}\int\limits_{x=\frac{1}{2}a+1}^{\xi} d\sin(ax-x^2) = O(1)$$

by the second mean-value theorem. Similarly, $J_1 = O(1)$. The conditions of Theorem 39 are thus satisfied.

Next, combine the cases $\alpha = 1$ and $\alpha = 3$ of (7.12.1). We obtain

$$\int_0^{\to\infty} \cos x^3 \cos ax\, dx = \frac{1}{6\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s) \cos \tfrac{1}{2}s\pi\, \Gamma(\tfrac{1}{3}-\tfrac{1}{3}s) \cos \tfrac{1}{6}\pi(1-s) a^{-s}\, ds.$$

Now

$$\Gamma(s)\Gamma(\tfrac{1}{3}-\tfrac{1}{3}s) = \frac{3^{s-\frac{1}{2}}}{2\pi} \Gamma(\tfrac{1}{3}s)\Gamma(\tfrac{1}{3}s+\tfrac{1}{3})\Gamma(\tfrac{1}{3}s+\tfrac{2}{3})\Gamma(\tfrac{1}{3}-\tfrac{1}{3}s)$$

$$= \frac{3^{s-\frac{1}{2}}}{2} \Gamma(\tfrac{1}{3}s)\Gamma(\tfrac{1}{3}s+\tfrac{1}{3})/\sin \tfrac{1}{3}\pi(1-s).$$

We thus obtain

$$\frac{1}{24\pi i} \int_{k-i\infty}^{k+i\infty} 3^{s-\frac{1}{2}}\{1+2\cos \tfrac{1}{3}\pi(1-s)\}\Gamma(\tfrac{1}{3}s)\Gamma(\tfrac{1}{3}s+\tfrac{1}{3}) a^{-s}\, ds.$$

In the first part put $s = \frac{3}{2}s'$. We obtain

$$\frac{1}{16\pi i} \int_{\frac{2}{3}k-i\infty}^{\frac{2}{3}k+i\infty} 3^{\frac{3}{2}s'-\frac{1}{2}}\Gamma(\tfrac{1}{2}s')\Gamma(\tfrac{1}{2}s'+\tfrac{1}{3}) a^{-\frac{3}{2}s'}\, ds'$$

$$= \frac{1}{4\pi i 2^{\frac{1}{3}} 3^{\frac{1}{2}}} \int_{\frac{2}{3}k-i\infty}^{\frac{2}{3}k+i\infty} 2^{s'+\frac{1}{3}-2}\Gamma(\tfrac{1}{2}s')\Gamma(\tfrac{1}{2}s'+\tfrac{1}{3}) \left(\frac{2a^{\frac{3}{2}}}{3\sqrt{3}}\right)^{-s'} ds' = \frac{a^{\frac{1}{2}}}{6} K_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\},$$

by (7.9.12). In the second part put $s = \frac{3}{2}s'-\frac{1}{2}$. We obtain

$$\frac{1}{8\pi i} \int_{\frac{2}{3}k+\frac{1}{3}-i\infty}^{\frac{2}{3}k+\frac{1}{3}+i\infty} 3^{\frac{3}{2}s'-1} \sin \tfrac{1}{2}\pi s'\, \Gamma(\tfrac{1}{2}s'-\tfrac{1}{6})\Gamma(\tfrac{1}{2}s'+\tfrac{1}{6}) a^{-\frac{3}{2}s'+\frac{1}{2}}\, ds'$$

$$= \frac{\sqrt{a}}{24\pi i} \int 2^{s'} \sin \tfrac{1}{2}\pi s'\, \Gamma(\tfrac{1}{2}s'-\tfrac{1}{6})\Gamma(\tfrac{1}{2}s'+\tfrac{1}{6})\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}^{-s'}\, ds'$$

$$= \frac{\pi\sqrt{a}}{6\sqrt{3}}[J_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}+J_{-\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}],$$

by (7.9.9). Hence†

$$\int_0^{\to\infty} \cos x^3 \cos ax\, dx = \frac{\pi\sqrt{a}}{6\sqrt{3}}[J_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}+J_{-\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}]+\frac{\sqrt{a}}{6} K_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}. \tag{7.12.5}$$

Similarly,

$$\int_0^{\to\infty} \sin x^3 \sin ax\, dx = \frac{\pi\sqrt{a}}{6\sqrt{3}}[J_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}+J_{-\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}]-\frac{\sqrt{a}}{6} K_{\frac{1}{3}}\{2(\tfrac{1}{3}a)^{\frac{3}{2}}\}. \tag{7.12.6}$$

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† Watson, § 6.4.

The process is justified by Theorem 39, as in the previous case. We have

$$\int\limits_{\lambda}^{\mu} x^{s-1}\cos x^{\alpha}\,dx = \frac{1}{\alpha}\int\limits_{\lambda'}^{\mu'} \xi^{s/\alpha-1}\cos\xi\,d\xi = O(|t|)$$

for $0 < \sigma/\alpha < 1$, as before. Also the integrals

$$\int\limits_{0}^{\to\infty} \cos(ax\pm x^3)\,dx$$

converge uniformly in any finite a-interval.

Again,

$$x^{\nu-1}\cos\frac{a^2}{x} = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(s)\cos\tfrac{1}{2}s\pi\, x^{\nu-1}\left(\frac{a^2}{x}\right)^{-s} ds$$

$$= \frac{1}{2\pi i}\int\limits_{1-\nu-k-i\infty}^{1-\nu-k+i\infty} \Gamma(1-\nu-s)\cos\tfrac{1}{2}\pi(1-\nu-s)\, a^{-2+2\nu+2s}x^{-s}\,ds.$$

Using this Mellin transformation and that of $\cos x$, we obtain†

$$\int\limits_{0}^{\infty} \cos x\cos\frac{a^2}{x}\,x^{\nu-1}\,dx = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(s)\cos\tfrac{1}{2}s\pi\,\Gamma(s-\nu)\cos\tfrac{1}{2}\pi(s-\nu)\,a^{2\nu-2s}ds$$

$$= \frac{1}{8\pi i}\int\limits_{2k-2\nu-i\infty}^{2k-2\nu+i\infty} \Gamma(\nu+\tfrac{1}{2}s')\Gamma(\tfrac{1}{2}s')\{\cos\tfrac{1}{2}\pi\nu+\cos\tfrac{1}{2}\pi(s'+\nu)\}\,a^{-s'}\,ds'$$

$$= \cos\tfrac{1}{2}\pi\nu\, a^{\nu}K_{\nu}(2a)+\frac{\pi a^{\nu}}{4\sin\frac{1}{2}\pi\nu}\{J_{-\nu}(2a)-J_{\nu}(2a)\}, \tag{7.12.7}$$

by (7.9.12) and (7.9.10). Similarly,

$$\int\limits_{0}^{\infty} \sin x\sin\frac{a^2}{x}\,x^{\nu-1}\,dx = \cos\tfrac{1}{2}\pi\nu\, a^{\nu}K_{\nu}(2a)-\frac{\pi a^{\nu}}{4\sin\frac{1}{2}\pi\nu}\{J_{-\nu}(2a)-J_{\nu}(2a)\}. \tag{7.12.8}$$

The process may be justified as in the previous cases.

7.13. Laplace transforms.

Simple functions $f(x)$, $\phi(s)$ connected by (1.4.1), (1.4.2) are

$$e^{-x}, \qquad \frac{1}{s+1}, \tag{7.13.1}$$

$$\cos x, \qquad \frac{s}{s^2+1}, \tag{7.13.2}$$

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† Watson, § 6.23.

$$\sin x, \qquad \frac{1}{s^2+1}, \tag{7.13.3}$$

$$x^{\alpha-1}, \qquad \Gamma(\alpha)s^{-\alpha}, \tag{7.13.4}$$

$$\frac{1}{\sqrt{x}}e^{-1/x}, \qquad \sqrt{\left(\frac{\pi}{s}\right)}e^{-2\sqrt{s}}. \tag{7.13.5}$$

The pair
$$x^\nu J_\nu(x), \qquad \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(1+s^2)^{\nu+\frac{1}{2}}} \tag{7.13.6}$$
results from (7.4.8); the pair
$$J_\nu(x), \qquad \frac{\{\sqrt{(1+s^2)}-s\}^\nu}{\sqrt{(1+s^2)}} \tag{7.13.7}$$
from (7.11.9); the pair
$$J_\nu(x)/x, \qquad \{\sqrt{(1+s^2)}-s\}^\nu/\nu \tag{7.13.8}$$
by integrating (7.11.9) with respect to a; the pair
$$x^{\frac{1}{2}\nu}J_\nu(\sqrt{x}), \qquad 2^{-\nu}s^{-\nu-1}e^{-1/(4s)} \tag{7.13.9}$$
comes from (7.11.10).

Writing
$$C(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^x \frac{\cos u}{\sqrt{u}}\,du, \qquad S(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^x \frac{\sin u}{\sqrt{u}}\,du, \tag{7.13.10}$$
we have
$$\int\limits_0^\infty e^{-xs}C(x)\,dx = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \frac{\cos u}{\sqrt{u}}\,du\int\limits_u^\infty e^{-xs}\,dx$$
$$= \frac{1}{\sqrt{(2\pi)}s}\int\limits_0^\infty \frac{\cos u}{\sqrt{u}}e^{-us}\,du = \frac{1}{2s}\left(\frac{1}{\sqrt{(1+s^2)}}+\frac{s}{1+s^2}\right)^{\frac{1}{2}}, \tag{7.13.11}$$
e.g. by (7.13.7). Similarly, we have the pair
$$S(x), \qquad \frac{1}{2s}\left(\frac{1}{\sqrt{(1+s^2)}}-\frac{s}{1+s^2}\right)^{\frac{1}{2}}. \tag{7.13.12}$$

Defining
$$\vartheta(x) = 1+2\sum_{n=1}^\infty e^{-n^2\pi^2x}, \tag{7.13.13}$$
we have
$$\int\limits_0^\infty e^{-xs}\vartheta(x)\,dx = \int\limits_0^\infty e^{-sx}\,dx+2\sum_{n=1}^\infty\int\limits_0^\infty e^{-(s+n^2\pi^2)x}\,dx$$
$$= \frac{1}{s}+2\sum_{n=1}^\infty\frac{1}{s+n^2\pi^2} = \frac{1}{\sqrt{s}\tanh\sqrt{s}}. \tag{7.13.14}$$

In each case the real part of s has to be greater than some lower limit —in fact 0 in all the above examples.

7.14. The formula (2.1.20) gives a number of interesting examples. The simplest is obtained by taking $f(x)$, $\phi(s)$ as in (7.13.4), and g, ψ with β for α. We obtain the familiar result

$$\int\limits_0^x y^{\alpha-1}(x-y)^{\beta-1}\,dy = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(\alpha)\Gamma(\beta)\frac{e^{xs}}{s^{\alpha+\beta}}\,ds$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1}. \tag{7.14.1}$$

The formula (2.1.20) is equivalent to Parseval's formula for the Fourier transforms

$$f(x)e^{-kx}\ (x>0),\ \ 0\ (x<0), \qquad (2\pi)^{-\frac{1}{2}}\phi(k+it),$$

and similarly with g and ψ. Here the e^{-kx} makes problems of f and g at infinity trivial. In the above case the L^2 theory applies if $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$. The result holds for $\alpha > 0$, $\beta > 0$. The extension may be made e.g. by Theorem 38.

The following examples are easily justified in a similar way.

Take $f(x)$ and $\phi(s)$ as in (7.13.6), but with parameter μ, and g, ψ, with ν. Then†

$$\int\limits_0^x y^{\mu}J_{\mu}(y)(x-y)^{\nu}J_{\nu}(x-y)\,dy$$

$$= \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{2^{\mu+\nu}\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\pi}\,\frac{e^{sx}}{(1+s^2)^{\mu+\nu+1}}\,ds$$

$$= \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\sqrt{(2\pi)}\Gamma(\mu+\nu+1)}x^{\mu+\nu+\frac{1}{2}}J_{\mu+\nu+\frac{1}{2}}(x). \tag{7.14.2}$$

The particular case $\mu = \frac{1}{2}$, $\nu = 0$ is

$$\int\limits_0^x \sin y\,J_0(x-y)\,dy = xJ_1(x). \tag{7.14.3}$$

A number of similar formulae derivable from this are given by Watson, § 12.21.

The particular case $\mu = 0$, $\nu = 0$ is

$$\int\limits_0^x J_0(y)J_0(x-y)\,dy = \sin x. \tag{7.14.4}$$

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† Hardy (10), Watson, §§ 12.2–12.22.

Similarly, (7.13.8) gives

$$\int_0^x \frac{J_\mu(y)J_\nu(x-y)}{y(x-y)}\,dy = \frac{1}{2\pi i\mu\nu}\int_{k-i\infty}^{k+i\infty}\{\sqrt{(1+s^2)}-s\}^{\mu+\nu}e^{xs}\,ds$$
$$= \frac{\mu+\nu}{\mu\nu}\frac{J_{\mu+\nu}(x)}{x}. \tag{7.14.5}$$

Taking f, ϕ as in (7.13.8), but with μ for ν, and g, ψ as in (7.13.7), we obtain

$$\int_0^x J_\mu(y)J_\nu(x-y)\,\frac{dy}{y} = \frac{1}{2\pi i\mu}\int_{k-i\infty}^{k+i\infty}\frac{\{\sqrt{(1+s^2)}-s\}^{\mu+\nu}}{\sqrt{(1+s^2)}}e^{xs}\,ds$$
$$= \frac{J_{\mu+\nu}(x)}{\mu}. \tag{7.14.6}$$

Taking g, ψ as in (7.13.7), and f, ϕ with μ for ν, we obtain

$$\int_0^x J_\mu(y)J_\nu(x-y)\,dy = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\{\sqrt{(1+s^2)}-s\}^{\mu+\nu}}{1+s^2}e^{xs}\,ds$$
$$= \frac{1}{\pi i}\int [\{\sqrt{(1+s^2)}-s\}^{\mu+\nu+1}-\{\sqrt{(1+s^2)}-s\}^{\mu+\nu+3}+\dots]\frac{e^{xs}}{\sqrt{(1+s^2)}}\,ds$$
$$= 2\{J_{\mu+\nu+1}(x)-J_{\mu+\nu+3}(x)+\dots\}. \tag{7.14.7}$$

The integral is expressible in finite terms if $\mu+\nu$ is an integer; for example

$$\int_0^x J_{-\nu}(y)J_\nu(x-y)\,dy = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{e^{xs}}{1+s^2}\,ds = \sin x. \tag{7.14.8}$$

A slightly different type of formula is obtained by taking

$$f(x) = x^{\frac{1}{2}\mu}J_\mu(a\sqrt{x}), \qquad \phi(s) = \frac{a^\mu}{2^\mu s^{\mu+1}}\exp\left(-\frac{a^2}{4s}\right),$$

and g, ψ with b, ν for a, μ. Then[†]

$$\int_0^x y^{\frac{1}{2}\mu}J_\mu(a\sqrt{y})(x-y)^{\frac{1}{2}\nu}J_\nu\{b\sqrt{(x-y)}\}\,dy$$
$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{a^\mu b^\nu}{2^{\mu+\nu}s^{\mu+\nu+2}}\exp\left(-\frac{a^2+b^2}{4s}+xs\right)ds$$
$$= \frac{2a^\mu b^\nu}{(a^2+b^2)^{\frac{1}{2}(\mu+\nu+1)}}J_{\mu+\nu+1}[\sqrt{\{(a^2+b^2)x\}}]. \tag{7.14.9}$$

[†] The result is equivalent to Watson, § 12.13 (1).

Taking the same g, ψ, and the f, ϕ of (7.13.4), we obtain[†]

$$\int\limits_0^x y^{\alpha-1}(x-y)^{\frac12\nu}J_\nu\{b\surd(x-y)\}\,dy = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{\Gamma(\alpha)b^\nu}{2^\nu s^{\nu+\alpha+1}}\exp\left(-\frac{b^2}{4s}+xs\right)ds$$

$$= \frac{2^\alpha\Gamma(\alpha)}{b^\alpha}x^{\frac12\nu}J_{\nu+\alpha}(b\surd x). \tag{7.14.10}$$

From (7.13.11) and (7.13.12) we obtain[‡]

$$\int\limits_0^x C(y)S(x-y)\,dy = \frac{1}{8\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{e^{xs}}{s^2(1+s^2)}\,ds$$

$$= \tfrac14(x-\sin x), \tag{7.14.11}$$

$$\int\limits_0^x \{C(y)C(x-y)-S(y)S(x-y)\}\,dy = \frac{1}{4\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{e^{xs}}{s(1+s^2)}ds$$

$$= \frac{1}{4\pi i}\int\limits_{k-i\infty}^{k+i\infty}\left(\frac1s-\frac{s}{1+s^2}\right)e^{xs}\,ds = \tfrac12(1-\cos x), \tag{7.14.12}$$

and there are some similar formulae involving J_0 and J_1.

The method also leads to an integral equation[||] satisfied by the function $\vartheta(x)$. From (7.13.14) we deduce

$$\int\limits_0^x \vartheta(y)\vartheta(x-y)\,dy = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{e^{xs}}{s\tanh^2\surd s}\,ds.$$

Now
$$\frac{d}{ds}\left(\frac{1}{\tanh\surd s}\right) = \frac{1}{2\surd s}-\frac{1}{2\surd s\tanh^2\surd s}.$$

Hence the right-hand side is

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{e^{xs}}{s}\,ds-\frac{1}{\pi i}\int\limits_{k-i\infty}^{k+i\infty}\left(\frac{d}{ds}\frac{1}{\tanh\surd s}\right)\frac{e^{xs}}{\surd s}\,ds$$

$$= 1+\frac{1}{\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{1}{\tanh\surd s}\left(\frac{x}{\surd s}-\frac{1}{2s^{\frac32}}\right)e^{xs}\,ds$$

$$= 1+2x\vartheta(x)-\int\limits_0^x\vartheta(u)\,du,$$

[†] Watson, § 12.11 (1).

[‡] Humbert (1).

[||] Due to F. Bernstein. See Hardy (10).

since $\displaystyle\int\limits_0^x \vartheta(u)\,du = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{e^{xs}}{s^{\frac12}\tanh\sqrt{s}}\,ds - \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{ds}{s^{\frac12}\tanh\sqrt{s}},$

and the last term is 0, as is seen by moving the line of integration away to the right. We have thus proved that

$$\int\limits_0^x \vartheta(y)\vartheta(x-y)\,dy = 1+2x\vartheta(x)-\int\limits_0^x \vartheta(u)\,du. \qquad (7.14.13)$$

VIII

GENERAL TRANSFORMATIONS

8.1. Generalization of Fourier's formulae. IN the foregoing chapters we have studied two formulae of the form

$$f(x) = \int_0^\infty k(xu)\,du \int_0^\infty k(uy)f(y)\,dy \tag{8.1.1}$$

for an arbitrary function $f(x)$; $k(x) = \sqrt{\left(\frac{2}{\pi}\right)}\cos x$ gives Fourier's cosine formula, and $k(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sin x$ gives Fourier's sine formula. There are, however, other formulae of the same form, the best known being Hankel's formula, in which

$$k(x) = x^{\frac{1}{2}}J_\nu(x). \tag{8.1.2}$$

There are also formulae of the form

$$f(x) = \int_0^\infty k(xu)\,du \int_0^\infty h(uy)f(y)\,dy \tag{8.1.3}$$

in which the two cosines of Fourier's formula are replaced by different functions. The simplest formula of this type is that in which

$$k(x) = x^{\frac{1}{2}}Y_\nu(x), \qquad h(x) = x^{\frac{1}{2}}\mathbf{H}_\nu(x). \tag{8.1.4}$$

As usual, (8.1.1) may be written as a pair of reciprocal formulae

$$g(x) = \int_0^\infty f(y)k(xy)\,dy, \tag{8.1.5}$$

$$f(x) = \int_0^\infty g(y)k(xy)\,dy. \tag{8.1.6}$$

A function $k(x)$ giving rise to a formula of the form (8.1.1) will be called a *Fourier kernel*. The main object of this chapter is to give an account of such functions.†

Suppose that we multiply (8.1.5) by x^{s-1} and integrate over $(0,\infty)$. We obtain formally

$$\int_0^\infty g(x)x^{s-1}\,dx = \int_0^\infty x^{s-1}\,dx \int_0^\infty f(y)k(xy)\,dy$$

$$= \int_0^\infty f(y)\,dy \int_0^\infty k(xy)x^{s-1}\,dx = \int_0^\infty f(y)y^{-s}\,dy \int_0^\infty k(u)u^{s-1}\,du,$$

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† Hardy and Titchmarsh (8), Watson (2).

i.e., with the usual notation for Mellin transforms,

$$\mathfrak{G}(s) = \mathfrak{F}(1-s)\mathfrak{K}(s). \tag{8.1.7}$$

Similarly (8.1.6) gives

$$\mathfrak{F}(s) = \mathfrak{G}(1-s)\mathfrak{K}(s). \tag{8.1.8}$$

Changing s into $1-s$ in one of these equations, and multiplying, we deduce that

$$\mathfrak{K}(s)\mathfrak{K}(1-s) = 1. \tag{8.1.9}$$

We should therefore expect that a Fourier kernel $k(x)$ would be in some sense of the form

$$k(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathfrak{K}(s)x^{-s}\,ds, \tag{8.1.10}$$

where $\mathfrak{K}(s)$ satisfies the functional equation (8.1.9).

8.2. The condition (8.1.9) may also be expected to be in some sense sufficient.

A characteristic property of a Fourier kernel $k(x)$ is that, if

$$k_1(x) = \int\limits_0^x k(u)\,du, \tag{8.2.1}$$

then

$$\int\limits_0^\infty k(xu)\frac{k_1(\xi u)}{u}\,du = \begin{matrix} 1 & (0 < x < \xi), \\ 0 & (x > \xi). \end{matrix} \tag{8.2.2}$$

For if we put $f(x) = 1$ $(0 < x \leqslant \xi)$, 0 $(x > \xi)$, in (8.1.1), we obtain (8.2.2); and conversely (8.2.2) leads to

$$\begin{aligned}\int\limits_0^x f(y)\,dy &= \int\limits_0^\infty f(y)\,dy \int\limits_0^\infty k(yu)\frac{k_1(xu)}{u}\,du \\ &= \int\limits_0^\infty \frac{k_1(xu)}{u}\,du \int\limits_0^\infty k(uy)f(y)\,dy,\end{aligned}$$

from which (8.1.1) follows by formal differentiation.

Now (8.1.10) gives formally

$$k_1(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathfrak{K}(s)\frac{x^{1-s}}{1-s}\,ds. \tag{8.2.3}$$

Hence, by a formal application of Parseval's formula for Mellin transforms, in the form (2.1.20),

$$\int_0^\infty k(xu)\frac{k_1(\xi u)}{u}\,du = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{K}(s)\frac{\mathfrak{K}(1-s)}{s}x^{-s}\xi^s\,ds \quad (c>0)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{\xi}{x}\right)^s\frac{ds}{s} = 1 \;\; (0<x<\xi), \quad 0 \;\; (x>\xi).$$

Hence (8.2.2), and so the Fourier formula, is formally a consequence of (8.1.9).

8.3. Similar analysis holds for the unsymmetrical formulae arising from (8.1.3). If we now write

$$f(x) = \int_0^\infty k(xu)g(u)\,du, \tag{8.3.1}$$

$$g(u) = \int_0^\infty h(uy)f(y)\,dy, \tag{8.3.2}$$

the relations between Mellin transforms are

$$\mathfrak{F}(s) = \mathfrak{G}(1-s)\mathfrak{K}(s), \tag{8.3.3}$$

$$\mathfrak{G}(s) = \mathfrak{F}(1-s)\mathfrak{H}(s), \tag{8.3.4}$$

and, eliminating $\mathfrak{F}$ and $\mathfrak{G}$,

$$\mathfrak{K}(s)\mathfrak{H}(1-s) = 1. \tag{8.3.5}$$

We may regard (8.3.2) as the solution of the integral equation (8.3.1) for the unknown function $g(u)$, the 'solving kernel' $h(x)$ being given by

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{H}(s)x^{-s}\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\mathfrak{K}(1-s)}\,ds.$$

8.4. Examples. Before proceeding farther we shall give a number of examples.

(1) If $\qquad \mathfrak{K}(s) = 2^{s-\frac{1}{2}}\dfrac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} \quad (\nu>-1),$

then $\qquad k(x) = x^{\frac{1}{2}}J_\nu(x),$

and the formula is that of Hankel; the cases $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ are Fourier's cosine and sine formulae.

If $-2<\nu<-1$, then

$$k(x) = x^{\frac{1}{2}}\left\{J_\nu(x) - \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+1)}\right\},$$

and generally, if $-m-1 < \nu < -m$, then

$$k(x) = x^{\frac{1}{2}}\left\{J_\nu(x) - \sum_{n=0}^{m} \frac{(-1)^n(\frac{1}{2}x)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}\right\},$$

the sum being the first $m+1$ terms in the power series for $J_\nu(x)$.

(2) If†
$$k(x) = x^{\frac{1}{2}}Y_\nu(x),$$
then, by (7.9.7),

$$\mathfrak{K}(s) = -2^{s-\frac{1}{2}}\pi^{-1}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4})\cos(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4})\pi.$$

This does not satisfy (8.1.9), so that $k(x)$ is not a Fourier kernel. But (8.3.5) gives

$$\mathfrak{H}(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}\tan(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\pi.$$

It then follows from (7.9.13) that

$$h(x) = x^{\frac{1}{2}}\mathbf{H}_\nu(x).$$

(3) There are a number of very general transformations, due to Fox,‡ in which $k(x)$ and $h(x)$ are linear combinations of generalized hypergeometric functions. From our present point of view these originate as follows.

Suppose that $\alpha_1 > 0$, that ρ_1 and ρ_2 are any real numbers other than negative integers, and that

$$\phi = \alpha_1-\rho_1-\rho_2+\tfrac{1}{2},$$

and let
$$\mathfrak{K}(s) = 2^{s-1}\frac{\Gamma(\alpha_1+\frac{1}{2}\phi-\frac{1}{2}s)\Gamma(\frac{1}{2}s-\frac{1}{2}\phi)}{\Gamma(\rho_1+\frac{1}{2}\phi-\frac{1}{2}s)\Gamma(\rho_2+\frac{1}{2}\phi-\frac{1}{2}s)}.$$

We deduce from the calculus of residues that

$$\begin{aligned} k(x) &= (\tfrac{1}{2}x)^{-\phi}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)}{\Gamma(\rho_1+n)\Gamma(\rho_2+n)}\frac{(-\frac{1}{4}x^2)^n}{n!} \\ &= (\tfrac{1}{2}x)^{-\phi}\,{}_1F_2(\alpha_1,\rho_1,\rho_2;-\tfrac{1}{4}x^2) \end{aligned}$$

in the usual hypergeometric series notation. If we now calculate $\mathfrak{H}(s)$ from (8.3.5), and then $h(x)$ by summation of residues, we find

$$h(x) = h_1(x)+h_2(x),$$

where

$$h_1(x) = \frac{\sin(\alpha_1-\rho_1)\pi}{\sin(\rho_2-\rho_1)\pi}(\tfrac{1}{2}x)^{2\rho_1+\phi-1}\,{}_1F_2(1-\alpha_1+\rho_1,\,1-\rho_2+\rho_1,\rho_1;-\tfrac{1}{4}x^2),$$

and $h_2(x)$ is derived from $h_1(x)$ by interchange of ρ_1 and ρ_2. The formulae thus obtained are those of Fox's Theorem 1, in the special case $p = 1$. In the general case $\mathfrak{K}(s)$ is a more complicated product of gamma-functions of the same type.

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† Titchmarsh (3).

‡ Fox (1).

The case $\alpha_1 = 1$, $\rho_1 = \frac{3}{2}$, $\rho_2 = \nu+\frac{3}{2}$ gives example (2) above. The case $\alpha_1 = 1$, $\rho_1 = a+1$, $\rho_2 = \nu+a+1$ gives a more general transformation found by Hardy[†] and discussed by Cooke.[‡] The case $\alpha_1 = \nu+\frac{3}{2}$, $\rho_1 = \nu+1$, $\rho_2 = 2\nu+1$ gives

$$k(x) = \tfrac{1}{2}\sqrt{\pi}\,\frac{d}{dx}\{xJ_\nu^2(\tfrac{1}{2}x)\}, \qquad h(x) = -\sqrt{\pi}\,J_\nu(\tfrac{1}{2}x)Y_\nu(\tfrac{1}{2}x),$$

a transformation due to Bateman.[||] The case $\alpha_1 = \nu+a+\frac{1}{2}$, $\rho_1 = \nu+a+1$, $\rho_2 = 2\nu+a+1$ gives a more general transformation due to Titchmarsh.[††] Fuller details concerning these transformations will be found in § 5.2 of Fox's paper.

If we take

$$\mathfrak{K}(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(a_1+\frac{1}{2}s)\Gamma(a_2+\frac{1}{2}s)\Gamma(a_3-\frac{1}{2}s)}{\Gamma(b_1+\frac{1}{2}s)\Gamma(b_2-\frac{1}{2}s)\Gamma(b_3-\frac{1}{2}s)},$$

where $$a_1+a_2+a_3+\tfrac{1}{2} = b_1+b_2+b_3,$$

we obtain examples of Fox's Theorem 2. For example, if

$$a_1 = \tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}, \qquad a_2 = \tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, \qquad a_3 = \tfrac{1}{2},$$
$$b_1 = 1, \qquad b_2 = \tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}, \qquad b_3 = \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2},$$

$k(x)$ and $h(x)$ are each combinations of two hypergeometric functions, and can be reduced to the forms

$$k(x) = \frac{\sqrt{\pi}}{2\sqrt{2}\sin\frac{1}{2}(\mu+\nu)\pi}x\{J_{-\mu}(\tfrac{1}{2}x)J_{-\nu}(\tfrac{1}{2}x)-J_\mu(\tfrac{1}{2}x)J_\nu(\tfrac{1}{2}x)\},$$

$$h(x) = \frac{\sqrt{\pi}}{2\sqrt{2}\cos\frac{1}{2}(\mu-\nu)\pi}\frac{d}{dx}\{x[J_\mu(\tfrac{1}{2}x)J_{-\nu}(\tfrac{1}{2}x)+J_{-\mu}(\tfrac{1}{2}x)J_\nu(\tfrac{1}{2}x)]\}.$$

(4) If $$k(x) = x^{\frac{1}{2}}\left\{Y_\nu(x)+\frac{2}{\pi}\cos a\pi\,K_\nu(x)\right\},$$

then, by (7.9.7) and (7.9.11), $\mathfrak{K}(s)$ is

$$\frac{2^{2s-1}\Gamma(\frac{1}{4}s+\frac{1}{4}\nu+\frac{1}{8})\Gamma(\frac{1}{4}s+\frac{1}{4}\nu+\frac{5}{8})\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8})\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{5}{8})}{\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8}+\frac{1}{2}a)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s-\frac{1}{2}a)\times\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8}-\frac{1}{2}a)\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s+\frac{1}{2}a)}.$$

In this case again $h(x)$ is the sum of two hypergeometric series. There are two interesting particular cases. If $\nu = 0$, $a = 1$, then

$$\mathfrak{K}(s) = -2^{2s-1}\left\{\frac{\Gamma(\frac{1}{4}s+\frac{1}{8})}{\Gamma(\frac{3}{8}-\frac{1}{4}s)}\right\}^2$$

[†] Hardy (13).

[‡] Cooke (1).

[||] Bateman (1), (2).

[††] Titchmarsh (6).

which satisfies (8.1.9), so that, by (7.9.7), (7.9.11),

$$k(x) = x^{\frac{1}{2}}\left\{Y_0(x)-\frac{2}{\pi}K_0(x)\right\}$$

is a Fourier kernel. If $\nu = 2$, $a = 0$, then

$$\mathfrak{K}(s) = 2^{2s-1}\left\{\frac{\Gamma(\frac{1}{4}s+\frac{9}{8})}{\Gamma(\frac{11}{8}-\frac{1}{4}s)}\right\}^2 \frac{\frac{1}{4}s-\frac{3}{8}}{\frac{1}{4}s+\frac{1}{8}},$$

and $\mathfrak{K}(s)\mathfrak{K}(1-s) = -1$, so that

$$k(x) = -h(x) = x^{\frac{1}{2}}\left\{Y_2(x)+\frac{2}{\pi}K_2(x)\right\}.$$

The formulae in this case are due to A. L. Dixon and Hardy.† Much more general formulae of a similar character have been obtained by Steen‡ and Kuttner.||

(5) If $$\mathfrak{K}(s) = e^{-ai(s-\frac{1}{2})^2} \quad (a > 0),$$

then $$\mathfrak{H}(s) = e^{ai(s-\frac{1}{2})^2}.$$

Taking $c = \frac{1}{2}$ in (8.1.10), we find that

$$k(x) = \frac{1}{2\pi\surd x}\int\limits_{-\infty}^{\infty} e^{ait^2}\cos(t\log x)\,dt = \frac{e^{\frac{1}{4}\pi i}}{2\surd(\pi a x)}e^{-i(\log x)^2/4a},$$

and $h(x)$ is the conjugate function. The Fourier formula thus obtained may be reduced by a change of variable to the exponential form of the ordinary Fourier formula. It is

$$f(x) = \frac{1}{4\pi a}\int\limits_0^{\infty} e^{-i(\log xu)^2/4a}\frac{du}{\surd(xu)}\int\limits_0^{\infty} e^{i(\log uy)^2/4a}\frac{f(y)}{\surd(uy)}\,dy;$$

and, if we put $a = \frac{1}{2}$, and

$$x = e^{\xi}, \qquad u = e^{\zeta}, \qquad y = e^{\eta}, \qquad g(\xi) = e^{\frac{1}{2}i\xi^2+\frac{1}{2}\xi}f(e^{\xi}),$$

we obtain $$g(\xi) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i\xi\zeta}\,d\zeta\int\limits_{-\infty}^{\infty} e^{i\zeta\eta}g(\eta)\,d\eta.$$

This formula is not included in our standard form, since the limits are $-\infty$ and ∞ instead of 0 and ∞.

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† Hardy (17); see also Hardy and Titchmarsh (8).

‡ Steen (1).

|| Kuttner (1).

(6) If
$$\mathfrak{K}(s) = e^{a(s-\frac{1}{2})^3} \quad (a > 0),$$
then $k(x)$ is a Fourier kernel. Taking $c = \frac{1}{2}$ in (8.1.10), we find

$$k(x) = \frac{1}{\pi\sqrt{x}} \int\limits_0^{\to\infty} \cos(at^3+t\log x)\,dt$$
$$= \frac{1}{3\sqrt{(3ax)}}\left(\log\frac{1}{x}\right)^{\frac{1}{2}}\left[J_{\frac{1}{3}}\left\{\frac{2}{3\sqrt{(3a)}}\left(\log\frac{1}{x}\right)^{\frac{3}{2}}\right\}+J_{-\frac{1}{3}}\left\{\frac{2}{3\sqrt{(3a)}}\left(\log\frac{1}{x}\right)^{\frac{3}{2}}\right\}\right]$$
$$(0 < x < 1),$$
$$= \frac{1}{3\pi\sqrt{(ax)}}(\log x)^{\frac{1}{2}}K_{\frac{1}{3}}\left\{\frac{2}{3\sqrt{(3a)}}(\log x)^{\frac{3}{2}}\right\} \quad (x > 1),$$
by (7.12.5) and (7.12.6).

(7) If
$$\mathfrak{K}(s) = \exp\{ie^{-i(s-\frac{1}{2})}\} = \exp(ie^t) \quad (s = \tfrac{1}{2}+it),$$
then
$$\mathfrak{H}(s) = \exp(-ie^{-t}).$$
We obtain
$$k(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp(ie^t)x^{-\frac{1}{2}-it}\,dt = \frac{x^{-\frac{1}{2}}}{2\pi}\int\limits_0^{\infty} e^{iu}u^{-i\log x}\,\frac{du}{u}$$
$$= \frac{x^{-\frac{1}{2}}}{2\pi}\int\limits_0^{\infty} e^{-v}(ve^{\frac{1}{2}i\pi})^{-i\log x}\,\frac{dv}{v} = \frac{x^{\frac{1}{2}\pi-\frac{1}{2}}}{2\pi}\Gamma(-i\log x),$$
and $h(x)$ is the conjugate.†

(8) If
$$\mathfrak{K}(s) = 1,$$
then (8.1.9) is satisfied, but the integral (8.1.10) is not convergent. If, however, we regard (8.2.3), with $0 < c < 1$, as the definition of $k_1(x)$, we have
$$k_1(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{x^{1-s}}{1-s}\,ds = \begin{matrix} 1 & (x > 1), \\ 0 & (0 < x < 1). \end{matrix}$$
If we replace (8.1.5), (8.1.6) by
$$g(x) = \frac{1}{x}\int\limits_0^{\infty} f(y)\,dk_1(xy), \qquad f(x) = \frac{1}{x}\int\limits_0^{\infty} g(y)\,dk_1(xy),$$
then our formulae become
$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right), \qquad f(x) = \frac{1}{x}g\left(\frac{1}{x}\right),$$
which are plainly consequences of one another.

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† Paley and Wiener, *Fourier Transforms*, § 16.

(9) In all these examples $\mathfrak{K}(s)$ is analytic. Suppose, however, that $c = \frac{1}{2}$, and

$$\mathfrak{K}(\tfrac{1}{2}+it) = i\operatorname{sgn} t.$$

Then $\mathfrak{H}$, defined by (8.3.5), is $-\mathfrak{K}$. The integral (8.1.10) is not convergent, but it is formally

$$\frac{i}{2\pi\sqrt{x}}\left(\int\limits_0^\infty x^{-it}\,dt - \int\limits_{-\infty}^0 x^{-it}\,dt\right) = \frac{1}{\pi\sqrt{x}}\int\limits_0^\infty \sin(t\log x)\,dt = \frac{1}{\pi\sqrt{x}\log x},$$

the integral being summable $(C, 1)$. Our formulae become

$$f(x) = \frac{1}{\pi}\int\limits_0^\infty \frac{g(y)\,dy}{\sqrt{(xy)}\log(xy)}, \qquad g(x) = -\frac{1}{\pi}\int\limits_0^\infty \frac{f(y)\,dy}{\sqrt{(xy)}\log(xy)}.$$

If we replace x and y by e^ξ and e^η, and interpret the integrals as principal values, we obtain formulae equivalent to those of the theory of Hilbert transforms.

(10) If
$$\mathfrak{K}(s) = \cot\tfrac{1}{2}s\pi,$$
then (8.1.9) is satisfied. The integral (8.1.10) is of the same type as in (9). A formal application of the theorem of residues gives

$$k(x) = \frac{2}{\pi}\frac{1}{1-x^2},$$

and we again obtain formulae of the Hilbert transform type.

(11) We obtain formulae of a somewhat different type by taking

$$\mathfrak{K}(\tfrac{1}{2}+it) = e^{i/t}.$$

Then (8.1.9) is satisfied, and

$$k(x) = \frac{1}{2\pi\sqrt{x}}\int\limits_{-\infty}^\infty e^{i/t}x^{-it}\,dt = \frac{1}{\pi\sqrt{x}}\int\limits_0^\infty \cos\left(\frac{1}{t}-t\log x\right)dt.$$

The integral is summable $(C, 1)$ if $x \neq 1$, and has the value

$$-\left(x\log\frac{1}{x}\right)^{-\frac{1}{2}}J_1\left\{2\left(\log\frac{1}{x}\right)^{\frac{1}{2}}\right\} \quad (0 < x < 1), \qquad 0 \quad (x > 1).$$

If $x = 1$, the integral for $k(x)$ diverges to infinity, and $k_1(x)$ has a discontinuity, as in example (8). The formula which results is therefore

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right) - \int\limits_0^{1/x} \frac{J_1\{2(-\log xy)^{\frac{1}{2}}\}}{(-xy\log xy)^{\frac{1}{2}}}f(y)\,dy.$$

If we put

$$x = e^{-\xi}, \qquad y = e^{-\eta}, \qquad e^{-\frac{1}{2}\xi}f(e^{-\xi}) = \phi(\xi), \qquad e^{-\frac{1}{2}\xi}g(e^{-\xi}) = \psi(\xi),$$

we obtain

$$\psi(\xi) = \phi(-\xi) - \int\limits_{-\xi}^{\infty} \frac{J_1\{2\sqrt{(\xi+\eta)}\}}{\sqrt{(\xi+\eta)}}\phi(\eta)\,d\eta.$$

The reciprocal formula is obtained by interchanging ϕ and ψ. The Fourier formula which results may be verified by using the integral†

$$\int\limits_{-\lambda}^{\infty} \frac{J_1\{2\sqrt{(x+\lambda)}\}J_1\{2\sqrt{(x+\mu)}\}}{\sqrt{(x+\lambda)}\sqrt{(x+\mu)}}\,dx = \frac{J_1\{2\sqrt{(\mu-\lambda)}\}}{\sqrt{(\mu-\lambda)}} \qquad (\lambda < \mu).$$

(12) The kernels which arise in the summation formulae obtained formally in § 2.9 are Fourier kernels. For example, in the argument of § 2.9 we obtain $2\cos 2\pi x$ and $4K_0(4\pi\sqrt{x})-2\pi Y_0(4\pi\sqrt{x})$ as the Mellin transforms of

$$\frac{\zeta(1-s)}{\zeta(s)}, \qquad \frac{\zeta^2(1-s)}{\zeta^2(s)}$$

respectively. These functions of course satisfy (8.1.9).

Note also that, if $k(x)$ is a Fourier kernel, so are $\sqrt{a}\,k(ax)$ and $\lambda x^{\frac{1}{2}(\lambda-1)}k(x^\lambda)$.

8.5. L^2-theory. In the theory of Fourier integrals we have proved theorems of two kinds, theorems on convergence in the ordinary sense, and theorems on mean-convergence. There are also theorems of both kinds for general transforms; but here the mean-convergence theory is both easier and more general than the other, and we begin with this.

In the first place, we need only assume the existence of the function $\mathfrak{K}(s)$ on the line $\sigma = \frac{1}{2}$. The equation (8.1.9) then takes the form

$$\mathfrak{K}(\tfrac{1}{2}+it)\mathfrak{K}(\tfrac{1}{2}-it) = 1. \qquad (8.5.1)$$

We might simply write $\mathfrak{K}(\frac{1}{2}+it) = \phi(t)$, and $\phi(t)\phi(-t) = 1$; but we shall retain the previous notation to preserve the appearance of the formulae.

We should now have formally

$$k(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathfrak{K}(\tfrac{1}{2}+it)x^{-\frac{1}{2}-it}\,dt. \qquad (8.5.2)$$

There is no reason in general to suppose that this integral will exist in any sense. However, the formula for $k_1(x)$ obtained by formal integration will exist in the sense that

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† Watson, § 13.47 (10).

$$k_1(x) = \frac{x}{2\pi} \operatorname*{l.i.m.}_{T\to\infty} \int\limits_{-T}^{T} \frac{\mathfrak{K}(\frac{1}{2}+it)}{\frac{1}{2}-it} x^{-\frac{1}{2}-it}\, dt. \tag{8.5.3}$$

If, as in most of our formulae, $\mathfrak{K}(s)$ takes conjugate values for conjugate values of s, then (8.5.1) gives

$$|\mathfrak{K}(\tfrac{1}{2}+it)| = 1. \tag{8.5.4}$$

Hence $\mathfrak{K}(\frac{1}{2}+it)/(\frac{1}{2}-it)$ belongs to $L^2(-\infty,\infty)$, the integral in (8.5.3) exists in the mean-square sense, and $k_1(x)/x$ belongs to $L^2(0,\infty)$.

It follows that our theorems have to be stated in terms, not of $k(x)$, but of $k_1(x)$. For example, (8.2.2) is no longer significant. However, the formula obtained by formal integration with respect to x is

$$\int\limits_0^\infty k_1(xu)k_1(\xi u)\,\frac{du}{u^2} = \min(x,\xi). \tag{8.5.5}$$

This integral is absolutely convergent in the general case, and (8.5.5) by itself may be taken as the basis of a Fourier theory.

The theory takes different forms according to whether (8.5.5) appears explicitly or not. The results may be summed up in the following theorems.

THEOREM 129. *Let $\mathfrak{K}(\frac{1}{2}+it)$ be any function of t satisfying* (8.5.1) *and* (8.5.4), *so that*

$$\frac{\mathfrak{K}(\frac{1}{2}+it)}{\frac{1}{2}-it}$$

belongs to $L^2(-\infty,\infty)$. Let $k_1(x)$ be defined by (8.5.3). *Let $f(x)$ be any function of $L^2(0,\infty)$. Then the formula*

$$g(x) = \frac{d}{dx}\int\limits_0^\infty k_1(xu)f(u)\,\frac{du}{u} \tag{8.5.6}$$

defines almost everywhere a function $g(x)$, also belonging to $L^2(0,\infty)$; the reciprocal formula

$$f(x) = \frac{d}{dx}\int\limits_0^\infty k_1(xu)g(u)\,\frac{du}{u} \tag{8.5.7}$$

also holds almost everywhere; and

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{g(x)\}^2\,dx. \tag{8.5.8}$$

THEOREM 130. *If $\mathfrak{K}(\frac{1}{2}+it)$ satisfies the conditions of Theorem* 129, *then $k_1(x)/x$ belongs to $L^2(0,\infty)$, and* (8.5.5) *holds.*

THEOREM 131. *Let $k_1(x)$ be such that $k_1(x)/x$ belongs to $L^2(0,\infty)$, and let* (8.5.5) *hold for all values of x and ξ. Then the reciprocal formulae of Theorem* 129 *hold.*

Theorem 129 is thus a consequence of Theorems 130 and 131. But it is possible to prove it directly.

The above theory is due to Watson.† We shall call functions $f(x)$ and $g(x)$ connected by (8.5.6), (8.5.7) k-transforms; and (8.5.8) the Parseval formula for k-transforms.

8.6. Proof‡ of Theorems 129, 130. Let $f(x)$ be any function of $L^2(0,\infty)$, and $\mathfrak{F}(s)$ its Mellin transform, so that $\mathfrak{F}(\frac{1}{2}+it)$ belongs to $L^2(-\infty,\infty)$. Since $|\mathfrak{K}(\frac{1}{2}+it)| = 1$, $\mathfrak{K}(\frac{1}{2}+it)\mathfrak{F}(\frac{1}{2}-it)$ also belongs to L^2. Let $g(x)$ be its Mellin transform. Then||

$$\int_0^x g(u)\,du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{K}(s)\mathfrak{F}(1-s)\frac{x^{1-s}}{1-s}\,ds.$$

Now $k_1(x)/x$ is the Mellin transform of $\mathfrak{K}(s)/(1-s)$. Hence, by the Parseval formula for Mellin transforms,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathfrak{K}(s)}{1-s}\mathfrak{F}(1-s)x^{-s}\,ds = \int_0^\infty \frac{k_1(xy)}{xy}f(y)\,dy.$$

Hence
$$\int_0^x g(u)\,du = \int_0^\infty \frac{k_1(xy)}{y}f(y)\,dy,$$

and (8.5.6) follows almost everywhere. The k-transform $g(x)$ of $f(x)$ is thus the Mellin transform of $\mathfrak{K}(s)\mathfrak{F}(1-s)$ (on $\sigma = \frac{1}{2}$). By the same rule, the k-transform of $g(x)$ is the Mellin transform of

$$\mathfrak{K}(s)\mathfrak{K}(1-s)\mathfrak{F}(s) = \mathfrak{F}(s).$$

Thus the k-transform of $g(x)$ is $f(x)$. All these transformations are of the class L^2, so that the necessary uniqueness theorems hold.

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† Watson (2). ‡ Busbridge (1).

|| This formula and the next come under Theorem 72, extended as in (2.1.23).

Finally,

$$\int_0^\infty \{g(x)\}^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{K}(\tfrac12+it)\mathfrak{F}(\tfrac12-it)|^2\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}(\tfrac12+it)|^2\,dt = \int_0^\infty \{f(x)\}^2\,dx,$$

the Parseval theorem for k-transforms.

Theorem 130 also follows at once from the Parseval formula for Mellin transforms. Since $k_1(x)/x$ is the Mellin transform of $\mathfrak{K}(s)/(1-s)$,

$$\int_0^\infty \frac{k_1(ax)}{ax}\,\frac{k_1(bx)}{bx}\,dx = \frac{1}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty} \frac{\mathfrak{K}(s)a^{-s}}{1-s}\,\frac{\mathfrak{K}(1-s)b^{s-1}}{s}\,ds$$

$$= \frac{1}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty} \frac{a^{-s}b^{s-1}}{(1-s)s}\,ds,$$

by the functional equation for $\mathfrak{K}(s)$. If $a > b$, the integral may be evaluated by moving the line of integration away to the right, and the value is a^{-1}. If $a > b$, its value, obtained by moving the line of integration to the left, is b^{-1}. Also the integral on the left is continuous at $a = b$. Hence the result.

8.7. Proof of Theorem 131.† Suppose first that $f(x)$ has a continuous derivative, and that it vanishes for all sufficiently small and sufficiently large values of x. Let

$$g_1(y) = \int_0^\infty \frac{k_1(xy)}{x} f(x)\,dx = \int_0^\infty \frac{k_1(u)}{u} f\left(\frac{u}{y}\right) du.$$

Then $g_1(y)$ is clearly differentiable, and

$$g(y) = g_1'(y) = -\frac{1}{y^2}\int_0^\infty k_1(u)f'\left(\frac{u}{y}\right) du = -\frac{1}{y}\int_0^\infty k_1(xy)f'(x)\,dx.$$

Hence

$$\int_0^\infty \{g(y)\}^2\,dy = \int_0^\infty \frac{dy}{y^2}\int_0^\infty k_1(xy)f'(x)\,dx \int_0^\infty k_1(\xi y)f'(\xi)\,d\xi$$

$$= \int_0^\infty f'(x)\,dx \int_0^\infty f'(\xi)\,d\xi \int_0^\infty \frac{k_1(xy)k_1(\xi y)}{y^2}\,dy$$

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† Titchmarsh (15); see also Plancherel (6).

$$= \int_0^\infty f'(x)\,dx \int_0^\infty f'(\xi)\min(x,\xi)\,d\xi$$

$$= \int_0^\infty f'(x)x\,dx \int_x^\infty f'(\xi)\,d\xi + \int_0^\infty f'(\xi)\xi\,d\xi \int_\xi^\infty f'(x)\,dx$$

$$= -2\int_0^\infty f(x)f'(x)x\,dx$$

$$= [-x\{f(x)\}^2]_0^\infty + \int_0^\infty \{f(x)\}^2\,dx$$

$$= \int_0^\infty \{f(x)\}^2\,dx.$$

All the transformations are easily justified if $f(x)$ satisfies the given conditions.

Next let $f(x)$ be any function of $L^2(0,\infty)$. Then it is known that there is a sequence of functions $f_n(x)$, each satisfying the conditions previously imposed on $f(x)$, and such that

$$\lim_{n\to\infty} \int_0^\infty \{f(x)-f_n(x)\}^2\,dx = 0.$$

Let $g_n(x)$ correspond to $f_n(x)$ in the same way as the above $g(x)$ does to $f(x)$. Then

$$\int_0^\infty \{g_m(x)-g_n(x)\}^2\,dx = \int_0^\infty \{f_m(x)-f_n(x)\}^2\,dx,$$

which tends to 0 as m and n tend to infinity. Hence the sequence $g_n(x)$ converges in mean, to a function $g(x)$ say. Then

$$\int_0^\infty \{g(x)\}^2\,dx = \lim_{n\to\infty}\int_0^\infty \{g_n(x)\}^2\,dx = \lim_{n\to\infty}\int_0^\infty \{f_n(x)\}^2\,dx = \int_0^\infty \{f(x)\}^2\,dx,$$

the Parseval formula.

Also

$$\int_0^y g(u)\,du = \lim_{n\to\infty}\int_0^y g_n(u)\,du$$

$$= \lim_{n\to\infty}\int_0^\infty \frac{k_1(xy)}{x} f_n(x)\,dx$$

$$= \int_0^\infty \frac{k_1(xy)}{x} f(x)\,dx,$$

so that
$$g(y) = \frac{d}{dy}\int_0^\infty \frac{k_1(xy)}{x} f(x)\,dx,$$
i.e. $g(y)$ is the k-transform of $f(x)$.

Let $\phi(x)$ be another function of $L^2(0,\infty)$, and $\psi(x)$ its k-transform. Then the Parseval formula gives
$$\int_0^\infty g(x)\psi(x)\,dx = \int_0^\infty f(x)\phi(x)\,dx.$$
Let
$$\phi(x) = 1 \ \ (x \leqslant u), \quad 0 \ \ (x > u).$$
Then
$$\int_0^\infty \frac{k_1(xy)}{x}\phi(x)\,dx = \int_0^u \frac{k_1(xy)}{x}\,dx = \int_0^{uy} \frac{k_1(x)}{x}\,dx,$$
and hence
$$\psi(y) = \frac{d}{dy}\int_0^{uy} \frac{k_1(x)}{x}\,dx = \frac{k_1(uy)}{y}.$$
Hence
$$\int_0^\infty g(x)\frac{k_1(ux)}{x}\,dx = \int_0^u f(x)\,dx,$$
and the reciprocal formula (8.5.7) follows.

8.8. Necessity of the conditions.† It is also easily seen that the conditions imposed on $k_1(x)$ and $\mathfrak{K}(s)$ in the above theorem are necessary. For suppose that the reciprocal formulae
$$\int_0^x f(y)\,dy = \int_0^\infty \frac{k_1(xu)}{u} g(u)\,du, \tag{8.8.1}$$
$$\int_0^x g(y)\,dy = \int_0^\infty \frac{k_1(xu)}{u} f(u)\,du \tag{8.8.2}$$
hold for any function $f(x)$ of $L^2(0,\infty)$. Let $f(x) = 1\ (x \leqslant \xi),\ 0\ (x > \xi)$. Then (8.8.2) gives
$$\int_0^x g(y)\,dy = \int_0^\xi \frac{k_1(xu)}{u}\,du = \int_0^{\xi x} \frac{k_1(v)}{v}\,dv,$$
so that
$$g(x) = \frac{k_1(\xi x)}{x}.$$

---

† Busbridge (1).

Substituting in (8.8.1), we obtain

$$\int_0^\infty \frac{k_1(xu)k_1(\xi u)}{u^2}\,du = \min(x,\xi).$$

In particular ($x = \xi = 1$) $k_1(u)/u$ belongs to $L^2(0,\infty)$.

If $\mathfrak{K}(s)/(1-s)$ is the Mellin transform of $k_1(x)/x$, Parseval's formula for Mellin transforms gives

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathfrak{K}(s)a^{-s}}{1-s}\,\frac{\mathfrak{K}(1-s)b^{s-1}}{s}\,ds = \int_0^\infty \frac{k_1(ax)}{ax}\,\frac{k_1(bx)}{bx}\,dx = \frac{\min(a,b)}{ab}.$$

But also
$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{a^{-s}}{1-s}\,\frac{b^{s-1}}{s}\,ds = \frac{\min(a,b)}{ab}.$$

Hence (taking $b = 1$)

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1-\mathfrak{K}(s)\mathfrak{K}(1-s)}{(1-s)s}\,a^{-s}\,ds = 0$$

for all values of a. Since the integrand (as the product of functions of L^2) belongs to L, it follows from Theorem 32, p. 47, that it must be null, i.e. that
$$\mathfrak{K}(s)\mathfrak{K}(1-s) = 1.$$

8.9. The unsymmetrical formulae. For the transformation arising from the equation (8.3.5) a similar set of theorems holds. We now assume that $\mathfrak{H}(\frac{1}{2}+it)$ and $\mathfrak{K}(\frac{1}{2}+it)$ are both bounded. Let $h_1(x)/x$ and $k_1(x)/x$ be the Mellin transforms of $\mathfrak{H}(\frac{1}{2}+it)/(\frac{1}{2}-it)$ and $\mathfrak{K}(\frac{1}{2}+it)/(\frac{1}{2}-it)$. Then a given function $f(x)$ of $L^2(0,\infty)$ has two transforms

$$g_h(x) = \frac{d}{dx}\int_0^\infty \frac{h_1(xy)}{y}f(y)\,dy, \qquad g_k(x) = \frac{d}{dx}\int_0^\infty \frac{k_1(xy)}{y}f(y)\,dy.$$

The k-transform of $g_h(x)$ is $f(x)$, and so is the h-transform of $g_k(x)$. The usual Parseval formula is replaced by the relation

$$\int_0^\infty g_h(x)g_k(x)\,dx = \int_0^\infty \{f(x)\}^2\,dx,$$

together with the inequalities

$$\int_0^\infty |g_h(x)|^2\,dx < c\int_0^\infty \{f(x)\}^2\,dx,$$

$$\int_0^\infty |g_k(x)|^2\,dx < c\int_0^\infty \{f(x)\}^2\,dx.$$

The proof of Theorem 129 extends without substantial change to this case. That of 131 holds only if $\mathfrak{K}(\frac{1}{2}+it)$ and $\mathfrak{H}(\frac{1}{2}+it)$ are conjugate, so that $g_h(x)g_k(x) = |g_h(x)|^2$. In the general case the result still holds, but now we have to prove (8.3.5) as in § 8.8, and thence proceed as in the proof of Theorem 129.

8.10. A convergence theorem.

In the foregoing theory the transformation is expressed in terms of $k_1(x)$, which is not necessarily differentiable. To obtain the forms (8.1.5), (8.1.6) we require further restrictions, both on the kernel and on the function represented.†

THEOREM 132. *Suppose* (i) *that* $\mathfrak{K}(\frac{1}{2}+it)$ *satisfies* (8.5.1) *and* (8.5.4), *so that* $x^{-1}k_1(x)$, *defined by* (8.2.3) *with* $c = \frac{1}{2}$, *belongs to* $L^2(0,\infty)$; (ii) *that* $k_1(x)$ *is the integral of* $k(x)$; (iii) *that* $x^{-\frac{1}{2}}k_1(x)$ *is bounded.*

Let
$$f(x) = \frac{1}{x}\int\limits_0^x \phi(y)\,dy, \tag{8.10.1}$$
where $\phi(y)$ *belongs to* $L^2(0,\infty)$. *Then*
$$f(x) = \int\limits_{\to 0}^{\to\infty} k(xu)\,du \int\limits_{\to 0}^{\to\infty} k(uy)f(y)\,dy \tag{8.10.2}$$
for every positive x.

We have
$$|f(x)| \leqslant \frac{1}{x}\left\{\int\limits_0^x |\phi(y)|^2\,dy \int\limits_0^x dy\right\}^{\frac{1}{2}} = o(x^{-\frac{1}{2}})$$
as $x \to 0$; and
$$|f(x)| \leqslant \frac{1}{x}\int\limits_0^X |\phi(y)|\,dy + \frac{1}{x}\left\{\int\limits_X^\infty |\phi(y)|^2\,dy \int\limits_0^x dy\right\}^{\frac{1}{2}} = o(x^{-\frac{1}{2}})$$
as $x \to \infty$, by choosing first X and then x.

Let $\psi(x)$ be the k-transform of $\phi(x)$. Then $\psi(x)$ belongs to L^2, and
$$\int\limits_0^x \phi(y)\,dy = \int\limits_0^\infty \frac{k_1(xu)}{u}\psi(u)\,du.$$

Let
$$g(u) = \int\limits_u^\infty \frac{\psi(v)}{v}\,dv. \tag{8.10.3}$$

---

† Hardy and Titchmarsh (8); see also Morgan (2).

Then

$$\int_0^x \phi(y)\,dy = -\int_0^\infty k_1(xu)g'(u)\,du$$

$$= \lim_{\delta\to 0,\,\Delta\to\infty}\left\{-[k_1(xu)g(u)]_\delta^\Delta + x\int_\delta^\Delta k(xu)g(u)\,du\right\}. \qquad (8.10.4)$$

Now $$|g(u)| \leqslant \left\{\int_u^\infty |\psi(v)|^2\,dv \int_u^\infty \frac{dv}{v^2}\right\}^{\frac{1}{2}} = o\,(u^{-\frac{1}{2}})$$

as $u \to \infty$; and

$$|g(u)| \leqslant \int_\delta^\infty \left|\frac{\psi(v)}{v}\right| dv + \left\{\int_0^\delta |\psi(v)|^2\,dv \int_u^\infty \frac{dv}{v^2}\right\}^{\frac{1}{2}} = o\,(u^{-\frac{1}{2}})$$

as $u \to 0$, by choosing first δ and then u. Hence the integrated terms in (8.10.4) tend to 0, and we obtain

$$f(x) = \frac{1}{x}\int_0^x \phi(y)\,dy = \int_{\to 0}^{\to\infty} k(xu)g(u)\,du. \qquad (8.10.5)$$

Again, (8.10.3) may be written

$$g(u) = \int_0^\infty \psi(v)\mu(v)\,dv,$$

where $\mu(v) = 0\ (v < u)$, $1/v\ (v > u)$. Hence by the Parseval formula

$$g(u) = \int_0^\infty \phi(v)\lambda(v)\,dv,$$

where

$$\lambda(v) = \frac{d}{dv}\int_0^\infty \mu(t)\frac{k_1(vt)}{t}\,dt = \frac{d}{dv}\int_u^\infty \frac{k_1(vt)}{t^2}\,dt$$

$$= \frac{d}{dv}\left\{v\int_{uv}^\infty \frac{k_1(\xi)}{\xi^2}\,d\xi\right\} = \int_{uv}^\infty \frac{k_1(\xi)}{\xi^2}\,d\xi - \frac{k_1(uv)}{uv}.$$

Hence

$$g(u) = \int_0^\infty \phi(v)\,dv \int_{uv}^\infty \frac{k_1(\xi)}{\xi^2}\,d\xi - \int_0^\infty \phi(v)\frac{k_1(uv)}{uv}\,dv$$

$$= g_1(u) - g_2(u) \qquad (8.10.6)$$

say. Integrating by parts,

$$g_1(u) = \left[vf(v)\int_{vu}^\infty \frac{k_1(\xi)}{\xi^2}\,d\xi\right]_0^\infty + \int_{\to 0}^{\to\infty} vf(v)u\frac{k_1(uv)}{u^2v^2}\,dv, \qquad (8.10.7)$$

and the integrated terms vanish since $vf(v) = o(v^{\frac{1}{2}})$, and, since $k_1(\xi)/\xi$ is L^2,

$$\int\limits_{uv}^{\infty} \frac{k_1(\xi)}{\xi^2}\,d\xi = o\,(v^{-\frac{1}{2}})$$

as in the case of $g(u)$. Also

$$g_2(u) = \left[vf(v)\frac{k_1(uv)}{uv}\right]_0^{\infty} - \int\limits_{\to 0}^{\to\infty} vf(v)\frac{k(uv)}{v}\,dv + \int\limits_{\to 0}^{\to\infty} vf(v)\frac{k_1(uv)}{uv^2}\,dv, \tag{8.10.8}$$

and the integrated terms vanish since $v^{-\frac{1}{2}}k_1(uv) = O(1)$.

From (8.10.6), (8.10.7), and (8.10.8) it follows that

$$g(u) = \int\limits_{\to 0}^{\to\infty} k(uv)f(v)\,dv, \tag{8.10.9}$$

and (8.10.5) and (8.10.9) give the theorem.

8.11. The resultant of two Fourier kernels.† Let

$$m(x) = \int\limits_0^{\infty} k(xy)l(y)\,dy$$

be the resultant of $k(x)$ and $l(x)$. Then a formal rule is that, *if* $k(x)$ *and* $l(x)$ *are Fourier kernels, so is* $m(x)$. We may, for example, put

$$\int\limits_0^{\infty}\int\limits_0^{\infty} m(xu)m(ut)f(t)\,dudt = \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} k(xuy)k(utz)l(y)l(z)f(t)\,dudtdydz,$$

and the substitution $t = v/z$, $y = zw$ gives

$$\int\limits_0^{\infty}\int\limits_0^{\infty} l(z)l(zw)\,dzdw \int\limits_0^{\infty}\int\limits_0^{\infty} k(xzwu)k(uv)f\left(\frac{v}{z}\right)\,dudv$$

$$= \int\limits_0^{\infty}\int\limits_0^{\infty} l(z)l(zw)f(xw)\,dzdw = f(x)$$

if k and l are Fourier kernels.

We can also argue in terms of Mellin transforms. If $\mathfrak{K}$ and $\mathfrak{L}$ are the Mellin transforms of k and l, that of m is

$$\mathfrak{M}(s) = \int\limits_0^{\infty} m(x)x^{s-1}\,dx = \int\limits_0^{\infty} x^{s-1}\,dx \int\limits_0^{\infty} k(xy)l(y)\,dy$$

$$= \int\limits_0^{\infty} l(y)\,dy \int\limits_0^{\infty} k(xy)x^{s-1}\,dx = \int\limits_0^{\infty} l(y)y^{-s}\,dy \int\limits_0^{\infty} k(u)u^{s-1}\,du$$

$$= \mathfrak{L}(1-s)\mathfrak{K}(s).$$

---

† Hardy (20).

Hence $$\mathfrak{M}(s)\mathfrak{M}(1-s) = \mathfrak{K}(s)\mathfrak{K}(1-s)\mathfrak{L}(1-s)\mathfrak{L}(s) = 1,$$
and the result again follows. The argument is still of course purely formal.

The L^2 theory gives

THEOREM 133. *Let $k_1(x)$ and $l_1(x)$ satisfy the conditions of Theorem* 131, *and let $m_1(1/x)$ be the l-transform of $k_1(x)/x$. Then $m_1(x)$ also satisfies the conditions of Theorem* 131.

Here $m_1(1/x)$ is defined by
$$\int_0^x m_1\left(\frac{1}{u}\right) du = \int_0^\infty \frac{l_1(xu)}{u} \frac{k_1(u)}{u}\, du.$$
Now $m_1(a/x)$ is the l-transform of $k_1(ax)/x$. Hence by Parseval's formula for l-transforms
$$\int_0^\infty \frac{m_1(ax)m_1(bx)}{x^2}\, dx = \int_0^\infty m_1\left(\frac{a}{x}\right) m_1\left(\frac{b}{x}\right) dx$$
$$= \int_0^\infty \frac{k_1(ax)k_1(bx)}{x^2}\, dx = \min(a,b),$$
the required result.

As a particular case, let $s_1(x) = 0$ $(x < 1)$, 1 $(x \geqslant 1)$, so that
$$g(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \qquad f(x) = \frac{1}{x} g\left(\frac{1}{x}\right).$$
We call this the transformation S. If $k_1 = l_1$, then
$$\int_0^x m_1\left(\frac{1}{y}\right) dy = \int_0^\infty \frac{k_1(t)k_1(xt)}{t^2}\, dt = \min(1,x),$$
and $m_1 \equiv s_1$. If $l_1 = s_1$, then
$$\int_0^x m_1\left(\frac{1}{y}\right) dy = \int_{1/x}^\infty \frac{k_1(t)}{t^2}\, dt = \int_0^x k_1\left(\frac{1}{t}\right) dt,$$
and $m_1 \equiv k_1$. Thus the resultant of k and k is s, the resultant of k and s is k.

EXAMPLES. (1) If k and l are the cosine and sine transformations,
$$m_1(x) = \frac{1}{\pi}\log\left|\frac{x-1}{x+1}\right|, \qquad m(x) = \frac{2}{\pi}\frac{1}{1-x^2},$$

and the m-transformation is

$$g(x) = \frac{2}{\pi}\int\limits_0^\infty \frac{f(t)}{1-x^2t^2}\,dt.$$

If f is even, this gives

$$\frac{1}{x}g\left(\frac{1}{x}\right) = \frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{f(t)}{x-t}\,dt,$$

the Hilbert transform of $f(t)$.

The resultant of this transformation and k is defined by

$$m(x) = \frac{2}{\pi}\int\limits_0^\infty \frac{k(xt)}{1-t^2}\,dt,$$

or, regarding $k(x)$ as even, by

$$m(x) = \frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{k(t)}{x-t}\,dt.$$

Thus *the conjugate of a Fourier kernel is a Fourier kernel.*

(2) The function $l_1(x) = x$ $(x < 1)$, 0 $(x \geqslant 1)$, satisfies (8.5.5). We conclude that, if $k(x)$ is a Fourier kernel, then so is

$$m(x) = \int\limits_0^\infty k(xt)\,dl_1(t) = \int\limits_0^1 k(xt)\,dt - k(x) = \frac{1}{x}\int\limits_0^x k(u)\,du - k(x).$$

Similarly, taking $l_1(x) = 0$ $(x < 1)$, $\log x - 1$ $(x \geqslant 1)$, we find that

$$\int\limits_x^\infty \frac{k(u)}{u}\,du - k(x)$$

is a Fourier kernel.

(3) The resultant of $t^{\frac12}J_\nu(t)$ and $t^{-\frac12}J_{\nu-1}(1/t)$ is† $J_{2\nu-1}(2t^{\frac12})$.

(4) The resultant of $\sqrt{(2/\pi)}\cos x$, $\sqrt{(2/\pi)}x^{-1}\cos x^{-1}$ is (§ 7.12)

$$\frac{2}{\pi}\{K_0(2\sqrt{x}) - Y_0(2\sqrt{x})\};$$

that of $\sqrt{(2/\pi)}\sin x$, $\sqrt{(2/\pi)}x^{-1}\sin x^{-1}$ is $(2/\pi)\{K_0(2\sqrt{x}) + Y_0(2\sqrt{x})\}$.

The last kernel is also the conjugate of $J_0(2\sqrt{x})$.

(5) The resultant of $J_0(2\sqrt{x})$ and $\cos x$ is $-\sin x$, and that of $J_0(2\sqrt{x})$ and $\sin x$ is $\cos x$. This may be proved as in § 7.12.

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† Watson, § 13.61 (1), or as in § 7.12.

(6) We have

$$x^{\frac12}\int\limits_0^\infty tJ_\mu(xt)J_{-\mu}(t)\,dt = -\frac{2\sin\mu\pi}{\pi}\,\frac{x^{\mu+\frac12}}{1-x^2}\quad (C,1),$$

if $x \neq 1$, while when $x = 1$ the integral diverges like

$$\frac{\cos\mu\pi}{\pi}\int^\infty dt.$$

The divergence indicates that when we form the resultant of $x^{\frac12}J_\mu(x)$ and $x^{\frac12}J_{-\mu}(x)$, there will be a discontinuity in $m_1(x)$ at $x = 1$. In fact

$$m_1(x) = -\frac{2\sin\mu\pi}{\pi}\int\limits_0^x \frac{t^{\mu+\frac12}\,dt}{1-t^2}\quad (x<1),$$

$$-\frac{2\sin\mu\pi}{\pi}\int\limits_0^x \frac{t^{\mu+\frac12}\,dt}{1-t^2}+\cos\mu\pi\quad (x \geqslant 1).$$

The inversion formulae are

$$g(x) = -\frac{2\sin\mu\pi}{\pi}\int\limits_0^\infty \frac{(xt)^{\mu+\frac12}}{1-x^2t^2}f(t)\,dt+\cos\mu\pi\frac{1}{x}f\left(\frac{1}{x}\right)$$

and the reciprocal formula.

(7) If we form the resultant $m(x)$ of $\sqrt{(2/\pi)}\cos x$ and $J_{\frac14}(2\sqrt{x})$, and then replace $m(x)$ by $2^{-\frac12}x^{-1}m(1/2x)$, we obtain the Fourier kernel

$$(2x)^{\frac12}\{\cos(x-\tfrac18\pi)J_{\frac14}(x)+\sin(x-\tfrac18\pi)J_{-\frac14}(x)\}.$$

8.12. Convergence of k-integrals. We now leave the transform theory, and prove quite independently a theorem on convergence in the ordinary sense. To do this we have to make very special assumptions, and the theory is practically restricted to those examples in § 8.4 in which $\mathfrak{K}(s)$ is a product of Γ-functions. For such functions, however, we obtain a direct generalization of Theorem 3.

THEOREM 134.† *Let* $\mathfrak{K}(s)$ *be regular in a strip* $\sigma_1 < \sigma < \sigma_2$, *where* $\sigma_1 < 0$, $\sigma_2 > 1$, *except perhaps for a finite number of simple poles on the imaginary axis; and let* $\mathfrak{K}(s)$ *be of the forms*

$$\mathfrak{K}_0(s)\left\{\alpha+\frac{\beta}{s}+O\left(\frac{1}{|s|^2}\right)\right\},\qquad \mathfrak{K}_0(s)\left\{\gamma+\frac{\delta}{s}+O\left(\frac{1}{|s|^2}\right)\right\}$$

---

† Hardy and Titchmarsh (8).

for large positive and negative t respectively, where

$$\mathfrak{K}_0(s) = \Gamma(s)\cos\tfrac{1}{2}s\pi$$

is the Mellin transform of $\cos x$. *Let* $\mathfrak{K}(s)$ *satisfy* (8.1.9), *and let* $k(x)$ *be the Mellin transform of* $\mathfrak{K}(s)$.

Let $x > 0$, *and let* $f(y)$ *be* $L(0, \infty)$, *and be of bounded variation near* $y = x$. *Then*

$$\int\limits_0^{\to\infty} k(xu)\,du \int\limits_0^{\infty} k(uy)f(y)\,dy = \tfrac{1}{2}\{f(x+0)+f(x-0)\}. \tag{8.12.1}$$

The function $\mathfrak{K}_0(s)$ is regular in any strip $\sigma_1 < \sigma < \sigma_2$, except for a finite number of simple poles at points where $\sigma \leqslant 0$. If t is large and positive, then

$$\mathfrak{K}_0(\sigma+it) = Ct^{\sigma-\frac{1}{2}}e^{i(t\log t-t)}\left\{1+\frac{a}{t}+O\left(\frac{1}{t^2}\right)\right\},$$

where C and a are complex, and a depends on σ; and $\mathfrak{K}_0(\sigma-it)$ satisfies the conjugate formula.

The functions

$$\Gamma(s)\sin\tfrac{1}{2}s\pi, \qquad \Gamma(s)\frac{\cos\frac{1}{2}s\pi}{1-s}, \qquad \Gamma(s)\frac{\sin\frac{1}{2}s\pi}{2-s}$$

are the Mellin transforms of

$$\sin x, \qquad \frac{\sin x}{x}, \qquad \frac{\sin x - x\cos x}{x^2},$$

and are of the form

$$\mathfrak{K}_0(s)\left\{i\operatorname{sgn} t+O\left(\frac{1}{|s|^2}\right)\right\}, \qquad \mathfrak{K}_0(s)\left\{-\frac{1}{s}+O\left(\frac{1}{|s|^2}\right)\right\},$$

$$\mathfrak{K}_0(s)\left\{\frac{-i\operatorname{sgn} t}{s}+O\left(\frac{1}{|s|^2}\right)\right\},$$

for large t. If $\mathfrak{K}(s)$ satisfies the conditions of the theorem, we can find constants a_1, a_2, a_3, a_4, such that

$$\mathfrak{K}(s) = \mathfrak{K}^{(1)}(s)+\mathfrak{K}^{(2)}(s)+\mathfrak{K}^{(3)}(s),$$

where

$$\mathfrak{K}^{(1)}(s) = a_1\Gamma(s)\cos\tfrac{1}{2}s\pi+a_2\Gamma(s)\sin\tfrac{1}{2}s\pi,$$

$$\mathfrak{K}^{(2)}(s) = a_3\Gamma(s)\frac{\cos\frac{1}{2}s\pi}{1-s}+a_4\Gamma(s)\frac{\sin\frac{1}{2}s\pi}{2-s},$$

and $$\mathfrak{K}^{(3)}(s) = O\{|\mathfrak{K}_0(s)s^{-2}|\} = O(|t|^{\sigma-\frac{5}{2}})$$

for large s of the strip. Let $k^{(1)}(x),\ldots$ be the Mellin transforms of $\mathfrak{K}^{(1)}(s),\ldots$.

8.13. LEMMA α. *$k(x)$ is bounded for all positive x.*

This is true for $k^{(1)}(x)$ and $k^{(2)}(x)$, so that it is enough to prove that

$$k^{(3)}(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathfrak{K}^{(3)}(s)x^{-s}\,ds$$

is bounded. If $x \geqslant 1$, we take $c = 1+\delta$, where $0 < \delta < \frac{1}{2}$, $1+\delta < \sigma_2$. Since $\mathfrak{K}^{(3)}(s)$ is then $O(|s|^{\delta-\frac{1}{2}})$, $k^{(3)}(x)$ is bounded, and indeed is $O(x^{-1-\delta})$. If $0 < x \leqslant 1$, we take $c = -\delta$, where $\sigma_1 < -\delta < 0$. Then

$$k^{(3)}(x) = \frac{1}{2\pi i}\int\limits_{-\delta-i\infty}^{-\delta+i\infty} \mathfrak{K}^{(3)}(s)x^{-s}\,ds + \rho,$$

the latter term being the sum of the residues at any poles on the imaginary axis. It is plain that ρ is bounded, and the integral is bounded because $\mathfrak{K}^{(3)}(s) = O(|s|^{-\delta-\frac{1}{2}})$. Hence $k(x)$ is bounded for all positive x.

8.14. LEMMA β. *Let*

$$\phi(\lambda, x, y) = \int\limits_{1/\lambda}^{\lambda} k(xu)k(yu)\,du, \tag{8.14.1}$$

where $\lambda > 1$ and x is positive and fixed. Then

$$|\phi| < B(x, \zeta) \tag{8.14.2}$$

for all positive y for which $|y-x| \geqslant \zeta$.

In view of Lemma α we may replace ϕ by

$$\begin{aligned}\chi(\lambda, x, y) &= \int\limits_{1}^{\lambda} k(xu)k(yu)\,du\\ &= \int\limits_{1}^{\lambda} \{k^{(1)}(xu)+k^{(2)}(xu)\}\{k^{(1)}(yu)+k^{(2)}(yu)+k^{(3)}(yu)\}\,du +\\ &\qquad\qquad + \int\limits_{1}^{\lambda} k^{(3)}(xu)k(yu)\,du.\end{aligned}$$

The last term is bounded because $k^{(3)}(xu) = O(u^{-1-\delta})$ and $k(yu)$ is bounded.

Denote the integral involving $k^{(p)}(xu)k^{(q)}(yu)$ by $\chi_{p,q}$. Then $\chi_{1,1}$ is clearly bounded. Next, $\chi_{1,2}$ splits up into four terms, a typical term being

$$\int_1^{\lambda} \sin xu \frac{\sin yu - yu\cos yu}{y^2u^2}\,du = \int_1^{1/y} \sin xu \frac{\sin yu - yu\cos yu}{y^2u^2}\,du +$$

$$+\int_{1/y}^{\lambda} \frac{\sin xu \sin yu}{y^2u^2}\,du - \int_{1/y}^{\lambda} \frac{\sin xu\,\cos yu}{yu}\,du.$$

Since $(\sin x - x\cos x)/x^2$ is positive increasing in $0 < x < 1$, the first term on the right is

$$(\sin 1 - \cos 1)\int_{u_1}^{1/y} \sin xu\,du,$$

where $0 < u_1 < 1/y$. The second and third are

$$\int_{1/y}^{u_2} \sin xu \sin yu\,du, \qquad -\int_{1/y}^{u_3} \sin xu \cos yu\,du,$$

where $u_2 > 1/y$, $u_3 > 1/y$. All these are bounded, and the other terms of $\chi_{1,2}$ may be shown to be bounded in the same way. Hence $\chi_{1,2}$ is bounded.

A similar argument applies to $\chi_{2,1}$ and $\chi_{2,2}$. Thus a typical term of $\chi_{2,2}$ is

$$\int_1^{\lambda} \frac{\sin xu}{xu}\frac{\sin yu}{yu}\,du$$

$$= \int_1^{1/y} + \int_{1/y}^{\lambda} = \frac{\sin y}{y}\int_1^{u_1} \frac{\sin xu}{xu}\,du + \int_{1/y}^{u_2} \frac{\sin xu}{xu} \sin yu\,du,$$

and each of these is bounded.

A typical term in $\chi_{1,3}$ is

$$\frac{1}{2\pi i}\int_1^{\lambda} \cos xu\,du \int_{c-i\infty}^{c+i\infty} (yu)^{-s}\mathfrak{K}^{(3)}(s)\,ds.$$

If $\mathfrak{K}^{(3)}$ has no pole on the imaginary axis, we may take $c = 0$, and invert, and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-it}\mathfrak{K}^{(3)}(it)\,dt \int_1^{\lambda} u^{-it}\cos xu\,du.$$

The inner integral is $O(\lambda) = O(t)$ for $t > \lambda$, while for $0 < t < \lambda$ it differs by $O(t)$ from

$$\int_t^\lambda \frac{\cos xu}{u^{it}}\,du = \left[\frac{\sin xu}{xu^{it}}\right]_t^\lambda - it\left[\frac{\cos xu}{x^2u^{it+1}}\right]_t^\lambda - \frac{it(it+1)}{x^2}\int_t^\lambda \frac{\cos xu}{u^{it+2}}\,du,$$

each term of which is $O(t)$. Since $\mathfrak{K}^{(3)}(it) = O(|t|^{-3})$ for large t, the term in question is bounded.

If there are poles on the imaginary axis, it is sufficient to consider one of them, say at $s = i\tau$ with residue C. Let

$$\mathfrak{K}^{(3)}(s) = C\Gamma(s-i\tau)+\mathfrak{K}^{(4)}(s), \qquad k^{(3)}(x) = Cx^{-i\tau}e^{-x}+k^{(4)}(x).$$

Then $\mathfrak{K}^{(4)}$ satisfies the conditions imposed above on $\mathfrak{K}^{(3)}$, and the additional term is

$$Cy^{-i\tau}\int_1^\lambda u^{-i\tau}e^{-yu}\cos xu\,du = Cy^{-i\tau}\int_1^{\lambda'} u^{-i\tau}\cos xu\,du = O(1)$$

by the argument used for the above inner integral. Hence $\chi_{1,3}$ is bounded. Practically the same argument proves that $\chi_{2,3}$ is bounded, and the lemma follows.

8.15. LEMMA γ. *Let*

$$\psi(\lambda,x,y) = \int_{1/\lambda}^{\lambda} k(xu)\frac{k_1(yu)}{u}\,du,$$

where
$$k_1(x) = \int_0^x k(u)\,du = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{K}(s)\frac{x^{1-s}}{1-s}\,ds.$$

Then $|\psi| < B(x,\zeta)$ *for* $\lambda > 1$, $x > 0$, *and* $0 < x-\zeta < y < x+\zeta$; *and* $\psi(\lambda,x,y)$ *converges* (*boundedly*) *as* $\lambda \to \infty$ *to the limit*

$$0 \quad (y < x), \qquad \tfrac{1}{2} \quad (y = x), \qquad 1 \quad (y > x).$$

Since $k(u) = O(1)$, $k_1(u) = O(u)$ for small u, the integral over $(1/\lambda, 1)$ is bounded. We now write

$$k(x) = k^{(1)}(x)+k^{(5)}(x),$$

where $k^{(1)}$ is the same as before; and

$$\int_1^\lambda k(xu)\frac{k_1(yu)}{u}\,du$$
$$= \int_1^\lambda k^{(1)}(xu)\frac{k_1^{(1)}(yu)}{u}\,du + \int_1^\lambda k(xu)\frac{k_1^{(5)}(yu)}{u}\,du + \int_1^\lambda k^{(5)}(xu)\frac{k_1^{(1)}(yu)}{u}\,du.$$

The first term is a multiple of

$$\int_1^\lambda (a_1\cos xu+a_2\sin xu)\frac{a_1\sin yu+a_2(1-\cos yu)}{u}\,du$$
$$=a_1^2\int_1^\lambda \frac{\cos xu\sin yu}{u}\,du+a_1a_2\int_1^\lambda \frac{\cos xu-\cos(x+y)u}{u}\,du+$$
$$+a_2^2\int_1^\lambda \frac{\sin xu(1-\cos yu)}{u}\,du,$$

each term of which converges boundedly. Also $k(u)$ and $k_1^{(1)}(u)$ are bounded,

$$k^{(5)}(u)=O\left(\int_{-\infty}^{\infty}(1+|t|)^{\sigma-\frac{1}{2}}u^{-\sigma}\,dt\right)=O(u^{-\frac{1}{2}+\delta}),$$

taking $\sigma=\frac{1}{2}-\delta$; and $k_1^{(5)}(u)$, like $uk^{(3)}(u)$, is $O(u^{-\delta})$. The remaining terms are therefore bounded.

This proves the lemma except as regards the value of the limit. To calculate this directly requires some further examination of the argument, but the result can be obtained from the transform theory. We have in fact proved that

$$\int_0^\infty \frac{k_1(u)k(xu)}{u}\,du$$

converges boundedly for $x\geqslant\delta$, where $0<\delta<1$, and uniformly except near $x=1$; hence, if its value is $\phi(x)$,

$$\int_\delta^x \phi(u)\,du=\int_0^\infty \frac{k_1(x)\{k_1(xu)-k_1(\delta u)\}}{u^2}\,du=\min(x,1)-\delta$$

and hence $\qquad \phi(x)=1 \quad (x<1), \qquad 0 \quad (x>1).$

If $x=1$,

$$\int_0^X \frac{k_1(u)k(u)}{u}\,du=\frac{k_1^2(X)}{X}-\int_0^X \frac{k(u)k_1(u)}{u}\,du+\int_0^X \frac{k_1^2(u)}{u^2}\,du,$$

and since each integral tends to a limit as $X\to\infty$, so does $k_1^2(X)/X$, and this limit must be zero since $k_1^2(X)/X^2$ belongs to $L(0,\infty)$. Hence

$$2\int_0^\infty \frac{k_1(u)k(u)}{u}\,du=\int_0^\infty \frac{k_1^2(u)}{u^2}\,du=1$$

8.16. The Riemann-Lebesgue theorem is here replaced by the following theorem, due to Hobson (1).

LEMMA δ. *Let $f(t)$ belong to $L(a,b)$; let $\phi(\lambda,t)$ belong to $L(a,b)$ for all values of λ, let it be bounded uniformly with respect to λ in (a,b); and let*

$$\int_{\alpha}^{\beta} \phi(\lambda,t)\,dt \to 0$$

as $\lambda \to \infty$, uniformly in α and β for $a \leqslant \alpha < \beta \leqslant b$. Then

$$\lim_{\lambda\to\infty} \int_a^b f(t)\phi(\lambda,t)\,dt = 0.$$

Suppose first that $f(t)$ is absolutely continuous in (a,b). Let

$$\int_a^t \phi(\lambda,u)\,du = \phi_1(\lambda,t).$$

Then $$\int_a^b f(t)\phi(\lambda,t)\,dt = f(b)\phi_1(\lambda,b) - \int_a^b f'(t)\phi_1(\lambda,t)\,dt.$$

Given ϵ, we have

$$|\phi_1(\lambda,t)| < \epsilon \qquad (\lambda > \lambda_0(\epsilon),\quad a \leqslant t \leqslant b),$$

and hence

$$\left|\int_a^b f(t)\phi(\lambda,t)\,dt\right| \leqslant \epsilon\left\{|f(b)| + \int_a^b |f'(t)|\,dt\right\} \quad (\lambda > \lambda_0).$$

The result therefore follows in this case.

In the general case we can, given ϵ, define an absolutely continuous function $\chi(t)$ such that

$$\int_a^b |f(t)-\chi(t)|\,dt < \epsilon.$$

Having fixed t and $\chi(t)$, we can, by the first part, choose λ_0 so large that

$$\left|\int_a^b \chi(t)\phi(\lambda,t)\,dt\right| < \epsilon.$$

If $|\phi(\lambda,t)| \leqslant M$, it follows that

$$\left|\int_a^b f(t)\phi(\lambda,t)\,dt\right| \leqslant \left|\int_a^b \chi(t)\phi(\lambda,t)\,dt\right| + \left|\int_a^b \{f(t)-\chi(t)\}\phi(\lambda,t)\,dt\right|$$
$$\leqslant \epsilon + M\epsilon \quad (\lambda > \lambda_0).$$

This proves the lemma.

8.17. Proof of Theorem 134. By Lemma α the integral

$$\int_0^\infty k(uy)f(y)\,dy$$

is uniformly convergent for $1/\lambda \leqslant u \leqslant \lambda$, so that we may multiply by $k(xu)$ and integrate under the integral sign. Hence

$$\begin{aligned}\int_{1/\lambda}^{\lambda} k(xu)\,du \int_0^\infty k(uy)f(y)\,dy &= \int_0^\infty f(y)\phi(\lambda,x,y)\,dy \\ &= \int_0^{\delta} + \int_{\delta}^{x-\zeta} + \int_{x-\zeta}^{x} + \int_{x}^{x+\zeta} + \int_{x+\zeta}^{\Delta} + \int_{\Delta}^{\infty} \\ &= I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}$$

It follows from Lemma β that

$$|I_1| \leqslant B\int_0^{\delta} |f(y)|\,dy < \epsilon, \qquad |I_6| < B\int_{\Delta}^{\infty} |f(y)|\,dy < \epsilon$$

for $\delta = \delta(\epsilon)$, $\Delta = \Delta(\epsilon)$, $\lambda > 2$.

Next

$$\begin{aligned}\int_{\alpha}^{\beta} \phi\,dy &= \int_{1/\lambda}^{\lambda} k(xu)\frac{k_1(\beta u)-k_1(\alpha u)}{u}\,du \\ &= \psi(\lambda,x,\beta)-\psi(\lambda,x,\alpha).\end{aligned}$$

If $\alpha<\beta<x$, or $x<\alpha<\beta$, this tends to 0, when $\lambda\to\infty$, by Lemma γ. Hence, by Lemma δ,

$$\lim_{\lambda\to\infty} I_2 = 0, \qquad \lim_{\lambda\to\infty} I_5 = 0$$

when ζ, δ, and Δ are fixed.

We may suppose ζ small enough to ensure that $f(y)$ is of bounded variation in $(x-\zeta, x+\zeta)$, and then

$$f(y)-f(x-0) = f_1(y)-f_2(y),$$

where f_1 and f_2 are positive and decreasing and tend to 0 when $y\to x$ from below. Then

$$\begin{aligned}I_3 &= f(x-0)\{\psi(\lambda,x,x)-\psi(\lambda,x,x-\zeta)\}+ \\ &\quad + \int_{x-\zeta}^{x} f_1(y)\phi(\lambda,x,y)\,dy - \int_{x-\zeta}^{x} f_2(y)\phi(\lambda,x,y)\,dy.\end{aligned}$$

The first term tends to $\frac{1}{2}f(x-0)$. The second is

$$f_1(x-\zeta)\int_{x-\zeta}^{\eta} \phi\,dy = f_1(x-\zeta)\{\psi(\lambda,x,\eta)-\psi(\lambda,x,x-\zeta)\},$$

where $x-\zeta < \eta < x$, and, since ψ is bounded, this is less than ϵ (for all λ in question) if ζ is sufficiently small. A similar argument applies to the third term. Hence

$$\left|\overline{\lim_{\lambda\to\infty}} I_3 - \tfrac{1}{2}f(x-0)\right| \leqslant 2\epsilon$$

if ζ is sufficiently small. There is a corresponding result for I_4, and it follows that

$$\lim_{\lambda\to\infty} \int\limits_{1/\lambda}^{\lambda} k(xu) \int\limits_{0}^{\infty} k(uy)f(y)\,dy = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

The u-integrand is, however, bounded as $u \to 0$, so that this may be replaced by (8.12.1). This proves the theorem.

It is easily verified that the $\mathfrak{K}(s)$ which gives rise to Hankel's theorem satisfies the above conditions if $\nu \geqslant -\frac{1}{2}$; and so do all the other $\mathfrak{K}$'s which are products of Γ-functions if the parameters involved are subject to suitable restrictions.

8.18. Hankel's theorem.†

The most important particular case of the foregoing theorem is that in which $k(x) = \sqrt{x}\,J_\nu(x)$. This case can be obtained much more simply.

THEOREM 135. *If $f(x)$ belongs to $L(0,\infty)$, and is of bounded variation near the point x, then for $\nu \geqslant -\frac{1}{2}$*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \int\limits_{0}^{\infty} J_\nu(xu)\sqrt{(xu)}\,du \int\limits_{0}^{\infty} J_\nu(uy)\sqrt{(uy)}f(y)\,dy. \tag{8.18.1}$$

Let δ be a small positive number. Then

$$\int\limits_{0}^{\lambda} J_\nu(xu)\sqrt{(xu)}\,du \int\limits_{0}^{x-\delta} J_\nu(uy)\sqrt{(uy)}f(y)\,dy$$

$$= \sqrt{x} \int\limits_{0}^{x-\delta} \sqrt{y}f(y)\,dy \int\limits_{0}^{\lambda} J_\nu(xu)J_\nu(uy)u\,du$$

$$= \sqrt{x}\,\lambda \int\limits_{0}^{x-\delta} \frac{xJ_{\nu+1}(\lambda x)J_\nu(\lambda y) - yJ_{\nu+1}(\lambda y)J_\nu(\lambda x)}{x^2-y^2}\sqrt{y}f(y)\,dy \tag{8.18.2}$$

$$= O(\sqrt{\lambda}) \int\limits_{0}^{x-\delta} J_\nu(\lambda y)\frac{\sqrt{y}f(y)}{x^2-y^2}\,dy + O(\sqrt{\lambda}) \int\limits_{0}^{x-\delta} J_{\nu+1}(\lambda y)\frac{y^{\frac{3}{2}}f(y)}{x^2-y^2}\,dy, \tag{8.18.3}$$

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† Watson, Chap. 14.

for any fixed x and δ. Now

$$\int\limits_0^{1/\lambda} J_\nu(\lambda y)\frac{\sqrt{y}f(y)}{x^2-y^2}\,dy = O\left(\int\limits_0^{1/\lambda} (\lambda y)^\nu y^{\frac{1}{2}}|f(y)|\,dy\right)$$

$$= O\left(\lambda^\nu \int\limits_0^{1/\lambda} y^{\nu+\frac{1}{2}}|f(y)|\,dy\right) = O\left(\lambda^{-\frac{1}{2}} \int\limits_0^{1/\lambda} |f(y)|\,dy\right) = o(\lambda^{-\frac{1}{2}}).$$

For $\lambda y \geqslant 1$ we have

$$J_\nu(\lambda y) = \frac{A\cos\lambda y + B\sin\lambda y}{(\lambda y)^{\frac{1}{2}}} + O\left(\frac{1}{(\lambda y)^{\frac{3}{2}}}\right).$$

The O-term contributes

$$O\left(\lambda^{-\frac{3}{2}} \int\limits_{1/\lambda}^{x-\delta} |f(y)|\,\frac{dy}{y}\right)$$

$$= O\left(\lambda^{-\frac{1}{2}} \int\limits_0^{1/\sqrt{\lambda}} |f(y)|\,dy\right) + O\left(\lambda^{-1} \int\limits_{1/\sqrt{\lambda}}^{x-\delta} |f(y)|\,dy\right) = o(\lambda^{-\frac{1}{2}}),$$

and the main term contributes

$$\lambda^{-\frac{1}{2}} \int\limits_{1/\lambda}^{x-\delta} (A\cos\lambda y + B\sin\lambda y)\,\frac{f(y)}{x^2-y^2}\,dy = o(\lambda^{-\frac{1}{2}})$$

by the Riemann-Lebesgue theorem. The second term in (8.18.3) may be dealt with in a similar way. Hence (8.18.2) tends to 0 as $\lambda \to \infty$.

Next, we may invert

$$\int\limits_0^{\lambda} J_\nu(xu)\sqrt{(xu)}\,du \int\limits_{x+\delta}^{\infty} J_\nu(uy)\sqrt{(uy)}f(y)\,dy$$

by the uniform convergence of the y-integral. The proof that this part tends to 0 is then similar, but simpler, since here y is not small.

We can suppose δ so small that $f(y)$ is of bounded variation over $(x-\delta, x+\delta)$. Then so is $y^{-\nu-\frac{1}{2}}f(y)$. Hence in $(x, x+\delta)$ we can write

$$y^{-\nu-\frac{1}{2}}f(y) = x^{-\nu-\frac{1}{2}}f(x+0)+\chi_1(y)-\chi_2(y),$$

where χ_1 and χ_2 are positive, increasing, and less than ϵ. Then

$$\int\limits_0^{\lambda} J_\nu(xu)\sqrt{(xu)}\,du \int\limits_{x}^{x+\delta} J_\nu(uy)\sqrt{(uy)}f(y)\,dy$$

$$= \sqrt{x}\int\limits_0^{\lambda} J_\nu(xu)u\,du \int\limits_{x}^{x+\delta} J_\nu(uy)y^{\nu+1}\{x^{-\nu-\frac{1}{2}}f(x+0)+\chi_1(y)-\chi_2(y)\}\,dy.$$

The first term in the bracket contributes

$$x^{-\nu}f(x+0)\int\limits_0^\lambda J_\nu(xu)[J_{\nu+1}\{(x+\delta)u\}(x+\delta)^{\nu+1}-J_{\nu+1}(xu)x^{\nu+1}]\,du$$
$$\to x^{-\nu}f(x+0)(x^\nu-\tfrac{1}{2}x^\nu) = \tfrac{1}{2}f(x+0),$$

by (7.11.15). The second term contributes

$$\sqrt{x}\int\limits_0^\lambda J_\nu(xu)u\,du\int\limits_x^{x+\delta} J_\nu(uy)y^{\nu+1}\chi_1(y)\,dy$$
$$= \sqrt{x}\int\limits_x^{x+\delta}\chi_1(y)y^{\nu+1}\,dy\int\limits_0^\lambda J_\nu(xu)J_\nu(yu)u\,du$$
$$= \sqrt{x}\,\chi_1(x+\delta)\int\limits_\xi^{x+\delta} y^{\nu+1}\,dy\int\limits_0^\lambda J_\nu(xu)J_\nu(yu)u\,du$$
$$= \sqrt{x}\,\chi_1(x+\delta)\int\limits_0^\lambda J_\nu(xu)[(x+\delta)^{\nu+1}J_{\nu+1}\{(x+\delta)u\}-\xi^{\nu+1}J_{\nu+1}(\xi u)]\,du$$

where $x < \xi < x+\delta$. Now for $x \geqslant x_0 > 0$, $y \geqslant x_0$,

$$\int\limits_0^\lambda J_\nu(xu)J_{\nu+1}(yu)\,du$$
$$= O(1)+\frac{2}{\pi}\int\limits_1^\lambda \cos(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\sin(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\,\frac{du}{\sqrt{(xy)u}}+\int\limits_1^\lambda O\left(\frac{1}{u^{\frac{3}{2}}}\right)du$$
$$= O(1)$$

for all λ. The contribution of the χ_1 term is therefore $O(\epsilon)$. Similarly so is that of the χ_2 term.

The theorem therefore follows on choosing first δ sufficiently small, and then, having fixed δ, λ sufficiently large.

8.19. Formulae derived from Hankel's theorem. Simple pairs of Hankel transforms may be derived from (7.4.6), (7.11.6), (7.11.8)–(7.11.15), and (7.11.17). Another elegant pair is

$$2^{3\nu-1}\Gamma(\nu+\tfrac{1}{2})\sqrt{\pi}\left(\frac{pq}{x}\right)^\nu\sqrt{x}\,J_\nu(px)J_\nu(qx),$$
$$\{x^2-(p-q)^2\}^{\nu-\frac{1}{2}}\{(p+q)^2-x^2\}^{\nu-\frac{1}{2}}x^{\frac{1}{2}-\nu} \quad (|p-q| < x < p+q),$$
$$0 \quad \text{elsewhere.} \qquad (8.19.1)$$

To prove this,† put $\nu = -\tfrac{1}{2}$, $\mu = \lambda-\tfrac{1}{2}$, $x = 1$, $y = \sin^2\theta$ in

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† This is Sonine's proof referred to by Watson, § 11.41.

(7.14.9). We obtain

$$a^{\lambda-\frac{1}{2}}\frac{J_\lambda\{\sqrt{(a^2+b^2)}\}}{(a^2+b^2)^{\frac{1}{2}\lambda}} = \frac{1}{\sqrt{(2\pi)}}\int_0^\pi \sin^{\lambda+\frac{1}{2}}\theta\, J_{\lambda-\frac{1}{2}}(a\sin\theta)\, e^{ib\cos\theta}\, d\theta. \tag{8.19.2}$$

Putting $a = q\sin\phi$, $b = p-q\cos\phi$, and multiplying by $\sin^{\lambda+\frac{1}{2}}\phi$ and integrating, we obtain

$$\int_0^\pi \frac{J_\lambda\{\sqrt{(p^2+q^2-2pq\cos\phi)}\}}{(p^2+q^2-2pq\cos\phi)^{\frac{1}{2}\lambda}}\sin^{2\lambda}\phi\, d\phi$$

$$= \frac{q^{\frac{1}{2}-\lambda}}{\sqrt{(2\pi)}}\int_0^\pi \sin^{\lambda+\frac{1}{2}}\phi\, d\phi \int_0^\pi \sin^{\lambda+\frac{1}{2}}\theta\, J_{\lambda-\frac{1}{2}}(q\sin\theta\sin\phi)\, e^{i\cos\theta(p-q\cos\phi)}\, d\theta$$

$$= \frac{q^{\frac{1}{2}-\lambda}}{\sqrt{(2\pi)}}\int_0^\pi \sin^{\lambda+\frac{1}{2}}\theta\, e^{ip\cos\theta}\, d\theta \int_0^\pi \sin^{\lambda+\frac{1}{2}}\phi\, J_{\lambda-\frac{1}{2}}(q\sin\theta\sin\phi)\, e^{-iq\cos\theta\cos\phi}\, d\phi$$

$$= q^{-\lambda}\int_0^\pi \sin^{\lambda+\frac{1}{2}}\theta\, e^{ip\cos\theta}\sin^{\lambda-\frac{1}{2}}\theta\, J_\lambda(q)\, d\theta$$

$$= 2^\lambda\Gamma(\lambda+\tfrac{1}{2})\sqrt{\pi}(pq)^{-\lambda}J_\lambda(p)J_\lambda(q). \tag{8.19.3}$$

The result stated follows on taking $p^2+q^2-2pq\cos\phi = \xi$ as a new variable.

The reciprocal formula† is

$$\int_0^\infty x^{1-\nu}J_\nu(px)J_\nu(qx)J_\nu(ux)\, dx$$

$$= \frac{\{u^2-(p-q)^2\}^{\nu-\frac{1}{2}}\{(p+q)^2-u^2\}^{\nu-\frac{1}{2}}}{2^{3\nu-1}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})(pqu)^\nu} \tag{8.19.4}$$

if $|p-q| < u < p+q$, and 0 otherwise.

Still other results can now be deduced from the Parseval formula.‡ For example, (7.11.12) gives the Hankel transforms

$$x^{\lambda+\nu+\frac{1}{2}}K_\lambda(ax), \qquad 2^{\lambda+\nu}a^\lambda x^{\nu+\frac{1}{2}}\Gamma(\lambda+\nu+1)(a^2+x^2)^{-\lambda-\nu-1}, \tag{8.19.5}$$

and we deduce

$$\int_0^\infty x^{\lambda+\mu+2\nu+1}K_\lambda(ax)K_\mu(bx)\, dx$$

$$= 2^{\lambda+\mu+2\nu}a^\lambda b^\mu\Gamma(\lambda+\nu+1)\Gamma(\mu+\nu+1)\int_0^\infty \frac{x^{2\nu+1}\, dx}{(a^2+x^2)^{\lambda+\nu+1}(b^2+x^2)^{\mu+\nu+1}}.$$

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† See Watson, § 13.46; Nicholson (1).

‡ See Titchmarsh (11).

We can put $x = b\tan\theta$ and expand the integral in powers of $(b^2-a^2)/b^2$; if $a = b$, the result (with $\nu = \frac{1}{2}\rho-\frac{1}{2}\lambda-\frac{1}{2}\mu-1$) is

$$\int\limits_0^\infty K_\lambda(ax)K_\mu(ax)x^{\rho-1}\,dx$$

$$= \frac{2^{\rho-3}}{\Gamma(\rho)a^\rho}\Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right) \tag{8.19.6}$$

Similarly, from the Hankel transforms (see (7.11.14))

$$\frac{x^{\nu+\frac{1}{2}}\sqrt{\pi}}{2^{3\nu}a^{2\nu}\Gamma(\nu+\frac{1}{2})}J_\nu(ax)K_\nu(ax), \qquad \frac{x^{\nu+\frac{1}{2}}}{(x^4+4a^4)^{\nu+\frac{1}{2}}}, \tag{8.19.7}$$

we deduce

$$\int\limits_0^\infty J_\nu^2(ax)K_\nu^2(ax)x^{2\nu+1}\,dx = \frac{2^{\nu-3}}{a^{2\nu+2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})\Gamma(\frac{3}{2}\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu+1)}, \tag{8.19.8}$$

and from (8.19.1), with $p = q$, we deduce

$$\int\limits_0^\infty J_\nu^4(px)x^{1-2\nu}\,dx = \frac{p^{2\nu-2}}{2}\frac{\Gamma(\nu)\Gamma(2\nu)}{\pi\{\Gamma(\nu+\frac{1}{2})\}^2\Gamma(3\nu)}. \tag{8.19.9}$$

IX

SELF-RECIPROCAL FUNCTIONS

9.1. Formalities. In previous chapters we have noticed a number of functions which are their own Fourier cosine or sine transforms, i.e. functions $f(x)$ such that

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty f(y) \cos xy \, dy \tag{9.1.1}$$

or

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty f(y) \sin xy \, dy. \tag{9.1.2}$$

The simplest solutions of (9.1.1) are

$$x^{-\frac{1}{2}}, \qquad e^{-\frac{1}{2}x^2}, \qquad \operatorname{sech}\{x\sqrt{(\tfrac{1}{2}\pi)}\}.$$

Similar solutions of (9.1.2) are

$$x^{-\frac{1}{2}}, \qquad xe^{-\frac{1}{2}x^2}, \qquad \frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}.$$

There are also functions which are their own Hankel transforms of order ν, i.e. solutions of

$$f(x) = \int\limits_0^\infty f(y)\sqrt{(xy)}J_\nu(xy) \, dy. \tag{9.1.3}$$

Solutions of (9.1.1), (9.1.2), (9.1.3) will be called R_c, R_s, R_ν respectively.

Other functions are 'skew-reciprocal', i.e. satisfy (9.1.1), (9.1.2), or (9.1.3) with the sign of the right-hand side changed. Such functions will be called $-R_c$, $-R_s$, $-R_\nu$ respectively.

The first problem of this chapter is to determine all self-reciprocal functions; or (since complete generality is hardly attainable) all such functions of certain classes, such as the class L^2. We shall take (9.1.1) as the typical case.

In a sense, there is an immediate solution. If $g(x)$ belongs to L^2, then $g(x)+G_c(x)$ is also a function of L^2, and is plainly self-reciprocal. Also any self-reciprocal $f(x)$ may be expressed as

$$\tfrac{1}{2}f(x)+\tfrac{1}{2}f(x) = \tfrac{1}{2}f(x)+\tfrac{1}{2}F_c(x).$$

The formula $g(x)+G_c(x)$ therefore gives the complete solution of the problem.

On the other hand, it is obvious that none of the examples quoted above have been obtained in this way, and the solution does not enable us to decide (unless by actual verification) whether a given $f(x)$ is self-reciprocal. To determine whether $f(x)$ is of the form $g+G_c$ is to solve another integral equation, viz.

$$f(x) = g(x)+\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty g(y)\cos xy\,dy. \tag{9.1.4}$$

We shall consider such equations in § 11.15; but it is easier to attack (9.1.1) directly.

Let $\mathfrak{F}(s)$ be the Mellin transform of $f(x)$. Then (9.1.1) gives formally

$$\begin{aligned}\mathfrak{F}(s) &= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty x^{s-1}\,dx\int_0^\infty f(y)\cos xy\,dy\\ &= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\,dy\int_0^\infty x^{s-1}\cos xy\,dx\\ &= \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(s)\cos\tfrac{1}{2}s\pi\int_0^\infty f(y)y^{-s}\,dy,\end{aligned}$$

i.e. $\mathfrak{F}(s)$ satisfies the functional equation

$$\mathfrak{F}(s) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(s)\cos\tfrac{1}{2}s\pi\,\mathfrak{F}(1-s). \tag{9.1.5}$$

If now we write $\mathfrak{F}(s) = 2^{\frac{1}{2}s}\Gamma(\frac{1}{2}s)\psi(s)$, then

$$\psi(s) = \psi(1-s), \tag{9.1.6}$$

and, by Mellin's formula,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\psi(s)x^{-s}\,ds. \tag{9.1.7}$$

We may therefore expect (9.1.7), where $\psi(s)$ satisfies (9.1.6), i.e. is an even function of $s-\frac{1}{2}$, to be a general formula for functions of R_c.

The simplest example is

$$\psi(s) = 1, \qquad f(x) = 2e^{-\frac{1}{2}x^2}.$$

We can deal with (9.1.2), or generally (9.1.3), in a similar way. If $f(x)$ satisfies (9.1.3), then

$$\mathfrak{F}(s) = \int_0^\infty y^{\frac{1}{2}}f(y)\,dy\int_0^\infty x^{s-\frac{1}{2}}J_\nu(xy)\,dx = \frac{2^{s-\frac{1}{2}}\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}\int_0^\infty f(y)y^{-s}\,dy,$$

i.e.
$$\mathfrak{F}(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}\mathfrak{F}(1-s). \tag{9.1.8}$$

Putting $\mathfrak{F}(s) = 2^{\frac{1}{2}s}\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})\psi(s)$, we obtain as a general solution of (9.1.3)
$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)x^{-s}\,ds, \tag{9.1.9}$$

where $\psi(s)$ again satisfies (9.1.6).

9.2. Another formal solution of the problem is obtained by considering
$$\chi(s) = \int\limits_0^\infty f(x)e^{-\frac{1}{2}s^2x^2}\,dx. \tag{9.2.1}$$

Then (9.1.1) gives
$$\chi(s) = \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty e^{-\frac{1}{2}s^2x^2}\,dx\int\limits_0^\infty f(y)\cos xy\,dy$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty f(y)\,dy\int\limits_0^\infty e^{-\frac{1}{2}s^2x^2}\cos xy\,dx$$
$$= \frac{1}{s}\int\limits_0^\infty f(y)e^{-\frac{1}{2}y^2/s^2}\,dy,$$

i.e.
$$\chi(s) = \frac{1}{s}\chi\left(\frac{1}{s}\right). \tag{9.2.2}$$

If $\mu(s) = s^{\frac{1}{4}}\chi(s^{\frac{1}{2}})$, then
$$\mu(s) = \mu\left(\frac{1}{s}\right). \tag{9.2.3}$$

We may write (9.2.1) as
$$\chi(s^{\frac{1}{2}}) = \int\limits_0^\infty (2u)^{-\frac{1}{2}}f\{(2u)^{\frac{1}{2}}\}e^{-su}\,du,$$

and the reciprocal formula is
$$(2u)^{-\frac{1}{2}}f\{(2u)^{\frac{1}{2}}\} = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \chi(s^{\frac{1}{2}})e^{su}\,ds,$$

or
$$f(x) = \frac{x}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mu(s)e^{\frac{1}{2}x^2s}s^{-\frac{1}{4}}\,ds. \tag{9.2.4}$$

Hence (9.2.4), where $\mu(s)$ satisfies (9.2.3), may be expected to be R_c.

The simplest example is

$$\mu(s) = 1, \qquad f(x) = \frac{2^{\frac{3}{4}}}{\Gamma(\frac{1}{4})} x^{-\frac{1}{2}}.$$

For general ν, let

$$\chi(s) = \int\limits_0^\infty f(x) x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}s^2x^2}\, dx. \tag{9.2.5}$$

Then (9.1.3) gives

$$\begin{aligned}\chi(s) &= \int\limits_0^\infty e^{-\frac{1}{2}s^2x^2} x^{\nu+\frac{1}{2}}\, dx \int\limits_0^\infty f(y)\sqrt{(xy)} J_\nu(xy)\, dy \\ &= \int\limits_0^\infty f(y)\sqrt{y}\, dy \int\limits_0^\infty e^{-\frac{1}{2}s^2x^2} x^{\nu+1} J_\nu(xy)\, dx \\ &= \int\limits_0^\infty f(y)\sqrt{y} \frac{y^\nu}{s^{2\nu+2}} e^{-y^2/2s^2}\, dy \\ &= \frac{1}{s^{2\nu+2}} \chi\left(\frac{1}{s}\right).\end{aligned}$$

If $\mu(s) = s^{\frac{1}{2}\nu+\frac{1}{2}} \chi(s^{\frac{1}{2}})$, then $\mu(s)$ again satisfies (9.2.3), and

$$f(x) = \frac{x^{\frac{1}{2}-\nu}}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \mu(s) e^{\frac{1}{2}x^2 s} s^{-\frac{1}{2}\nu-\frac{1}{2}}\, ds. \tag{9.2.6}$$

9.3. Still other formulae of the same kind can be obtained by replacing the $e^{-\frac{1}{2}x^2}$ of the above example by other functions which are self-reciprocal, and which also are the kernels of a general transformation. We may take, for example, the function

$$x^{\frac{1}{2}} J_{-\frac{1}{4}}(\tfrac{1}{2}x^2).$$

Proceeding as before, we obtain

$$\chi(s) = \int\limits_0^\infty (sx)^{\frac{1}{2}} J_{-\frac{1}{4}}(\tfrac{1}{2}s^2x^2) f(x)\, dx = \frac{1}{s} \chi\left(\frac{1}{s}\right),$$

and $f(x)$ can be expressed in terms of $\chi(s)$ by Hankel's theorem. This transformation has been studied in detail by Mehrotra (8).

9.4. Functions of L^2. We shall now justify the above arguments under a variety of conditions. The simplest conditions are provided by the L^2-theory of Mellin transforms.

THEOREM 136.[†] *A necessary and sufficient condition that a function $f(x)$ of $L^2(0,\infty)$ should be its own cosine transform is that it should*

[†] Hardy and Titchmarsh (4). Proof suggested by Miss Busbridge.

be of the form (9.1.7), *where* $c = \frac{1}{2}$, *the integral is a mean-square integral,*

$$\mathfrak{F}(\tfrac{1}{2}+it) = 2^{\frac{1}{4}+\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\psi(\tfrac{1}{2}+it) \tag{9.4.1}$$

belongs to $L^2(-\infty,\infty)$, *and* (9.1.6) *holds, i.e.* $\psi(\frac{1}{2}+it)$ *is an even function of* t.

In view of the L^2 theory of Mellin transforms, all that we have to prove is the equivalence of the self-reciprocal property of $f(x)$ to (9.1.6).

Now $f(y)$ and $\sin xy/y$ belong to $L^2(0,\infty)$, and the Mellin transform of the latter is

$$\Gamma(\tfrac{1}{2}+it)\cos\tfrac{1}{2}\pi(\tfrac{1}{2}+it)\,x^{\frac{1}{2}-it}/(\tfrac{1}{2}-it).$$

Hence

$$\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty f(y)\frac{\sin xy}{y}\,dy$$

$$= \frac{1}{2\pi}\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{-\infty}^{\infty} \mathfrak{F}(\tfrac{1}{2}-it)\Gamma(\tfrac{1}{2}+it)\cos\tfrac{1}{2}\pi(\tfrac{1}{2}+it)\frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt.$$

Also $$\int\limits_0^x f(y)\,dy = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathfrak{F}(\tfrac{1}{2}+it)\frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt.$$

If f is self-reciprocal, the right-hand sides must be equal; since each integrand belongs to $L(-\infty,\infty)$, they must be equal almost everywhere (Theorem 32, p. 47). Hence $\psi(\frac{1}{2}+it)$ is even. Conversely, if $\psi(\frac{1}{2}+it)$ is even, the right-hand sides are equal; hence so are the left-hand sides, and so f is self-reciprocal.

9.5. Functions of L^p.

THEOREM 137. *If a function* $f(x)$ *of* $L^p(0,\infty)$, *where* $1 < p < 2$, *is its own cosine transform, then it is of the form* (9.1.7), *where*

$$\mathfrak{F}(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\psi(s)$$

is an analytic function which (i) *is regular in the strip*

$$\frac{1}{p'} < \sigma < \frac{1}{p} \quad \left(p' = \frac{p}{p-1}\right), \tag{9.5.1}$$

(ii) *tends to* 0 *uniformly as* $s \to \infty$ *inside any interior strip, and* (iii) *satisfies* (9.1.5); *the integral in* (9.1.7) *is a mean-square integral along any line of the strip* (9.5.1).

This is a one-sided theorem, with conditions which are necessary only and not sufficient.

If $f(x)$ belongs to L^p, its cosine transform belongs to $L^{p'}$, so that $f(x)$ here belongs to both L^p and $L^{p'}$, and therefore to all intermediate classes L^q. In particular it belongs to L^2, and so satisfies the conditions of Theorem 136.

The function $\mathfrak{F}(s)$ reduces to the $\mathfrak{F}(\frac{1}{2}+it)$ of Theorem 136 when $s = \frac{1}{2}+it$, but $\mathfrak{F}(s)$ is now an analytic function, regular in the strip (9.5.1). In fact

$$\int_1^\infty |f(x)|x^{\sigma-1}\,dx$$
$$\leqslant \left(\int_0^1 |f|^{p'}\,dx\right)^{1/p'}\left(\int_0^1 x^{p(\sigma-1)}\,dx\right)^{1/p}+\left(\int_1^\infty |f|^p\,dx\right)^{1/p}\left(\int_1^\infty x^{p'(\sigma-1)}\,dx\right)^{1/p'},$$

and these integrals converge for the values of σ stated. It follows in the usual manner that $\mathfrak{F}(s)$ is regular in the strip, and bounded in any interior strip.

Again, we can write $\mathfrak{F}(s)$ in the form

$$\mathfrak{F}(s) = \left(\int_0^\delta + \int_\delta^\Delta + \int_\Delta^\infty\right) f(x)x^{\sigma-1}x^{it}\,dx = \mathfrak{F}_1(s)+\mathfrak{F}_2(s)+\mathfrak{F}_3(s).$$

Let $\eta > 0$, and

$$1/p'+\eta \leqslant \sigma \leqslant 1/p-\eta. \tag{9.5.2}$$

Then $$|\mathfrak{F}_1(s)| \leqslant \left(\int_0^\delta |f|^{p'}\,dx\right)^{1/p'}\left(\int_0^\delta x^{p(\sigma-1)}\,dx\right)^{1/p} = O(\delta^\eta)$$

as $\delta \to 0$, and we can therefore choose δ so that $|\mathfrak{F}_1| < \epsilon$ for all s in (9.5.2). Similarly, we can choose Δ so that $|\mathfrak{F}_3| < \epsilon$. When δ and Δ are fixed, $\mathfrak{F}_2 \to 0$ uniformly as $s \to \infty$ in (9.5.2). Hence $\mathfrak{F} \to 0$ uniformly in (9.5.2).

It follows from Theorem 136 that $\mathfrak{F}(s)$ satisfies (9.1.5) on $s = \frac{1}{2}+it$, and so throughout (9.5.1).

Thus $\mathfrak{F}(s)$ possesses the properties stated in the theorem, and it remains only to prove (9.1.7). This is true for $c = \frac{1}{2}$, by Theorem 136, so that it is sufficient to prove that the value of the integral is independent of c; and this follows by the argument of § 5.4.

9.6. The previous theorem is a one-sided theorem, and we cannot, in view of the asymmetry of the theory of transforms about the number 2, expect in this case a theorem as satisfactory as Theorem 136. There is, however, a very similar class of functions for which we can obtain a complete solution.

We shall say that $f(x)$ belongs to $L_p^*(0,\infty)$, where $1 < p < 2$, if $x^\alpha f(x)$ belongs to $L^2(0,\infty)$ for

$$-\alpha_0 = -\frac{1}{p}+\frac{1}{2} < \alpha < \frac{1}{p}-\frac{1}{2} = \alpha_0.$$

It is plain that $f(x)$ then belongs to $L^q(0,1)$ for $q \leqslant 2$. Suppose now that $p < q < 2$. Then we can choose $\alpha < \alpha_0$ so that $2q\alpha > 2-q$; and then

$$\int\limits_1^\infty |f|^q\,dx \leqslant \left(\int\limits_1^\infty x^{2\alpha}|f|^2\,dx\right)^{q/2}\left(\int\limits_1^\infty x^{-2q\alpha/(2-q)}\,dx\right)^{(2-q)/2} < \infty,$$

so that $f(x)$ belongs to L^q. If also $f(x)$ is its own cosine transform, it belongs to $L^{q'}$, so that a self-reciprocal $f(x)$ of L_p^* belongs to all L-classes between L^p and $L^{p'}$, though not usually to either of these.

The class of self-reciprocal functions of L_p^* is thus in this respect a little wider than the class of those of L^p. In other respects it is narrower. Suppose, for example, that $h(x)$ is defined by

$$h(x) = 2^{-n} \quad (n!-1 \leqslant x \leqslant n!+1,\ n = 2,3,\ldots),$$

and $h(x) = 0$ elsewhere. Then $h(x)$ belongs to L^r for every positive r, but to no L_p^*, since $\sum 2^{-rn}$ is convergent, but $\sum (n!)^{2\alpha}2^{-2n}$ divergent for every positive α. The cosine transform of $h(x)$ is

$$H_c(x) = \sqrt{\left(\frac{2}{\pi}\right)}\sum 2^{-n}\int\limits_{n!-1}^{n!+1}\cos xy\,dy = 2\sqrt{\left(\frac{2}{\pi}\right)}\frac{\sin x}{x}\sum\frac{\cos n!\,x}{2^n},$$

which is continuous and $O(x^{-1})$ at infinity, so that $H_c(x)$ belongs to L^r for $r > 1$, and to L_p^* for $1 < p < 2$. Thus $h(x)+H_c(x)$ is a self-reciprocal function which belongs to L^r for all $r > 1$, but to no L_p^*.

THEOREM 138. *A necessary and sufficient condition that a function* $f(x)$ *of* $L_p^*(0,\infty)$ *should be its own cosine transform is that it should be of the form* (9.1.7), *where* $\mathfrak{F}(s)$ *satisfies the conditions* (i), (ii), (iii) *of Theorem* 137, *and* (iv) *belongs to* $L^2(-\infty,\infty)$, *qua function of* t, *for all* σ *of* (9.5.1).

(i) *The condition is necessary.* Since $f(x)$ belongs to L^r for $p < r < p'$, we have only to show that $\mathfrak{F}(s)$ satisfies condition (iv). This results immediately from the theory of transforms, since

$$\mathfrak{F}(s) = \int\limits_0^\infty f(x)x^{\sigma-\frac{1}{2}}x^{-\frac{1}{2}+it}\,dx,$$

and $x^{\sigma-\frac{1}{2}}f(x)$ belongs to L^2 if $|\sigma-\frac{1}{2}| < \alpha_0$, i.e. if $1/p' < \sigma < 1/p$.

(ii) *The condition is sufficient.* Since $\mathfrak{F}(s)$ belongs to L^2 on the line $s = c+it$, the integral (9.1.7) exists as a mean-square integral for all c in question, and as before its value is independent of c. It therefore defines a function $f(x)$ independent of c. Since

$$x^{c-\frac{1}{2}}f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathfrak{F}(c+it)x^{-\frac{1}{2}-it}\,dt$$

for $|c-\frac{1}{2}| < \alpha_0$, and the right-hand side belongs to L^2 for every such c, $f(x)$ belongs to L_p^*. Finally, by Theorem 136, $f(x)$ is self-reciprocal.

9.7. Analytic functions. We shall say that $f(x)$ belongs to $A(\alpha, a)$, where $0 < \alpha \leqslant \pi$, $a < \frac{1}{2}$, if (i) it is an analytic function of $x = re^{i\theta}$ regular in the angle defined by $r > 0$, $|\theta| < \alpha$, and (ii) it is $O(|x|^{-a-\epsilon})$ for small x, and $O(|x|^{a-1+\epsilon})$ for large x, for every positive ϵ and uniformly in any angle $|\theta| \leqslant \alpha-\eta < \alpha$.

THEOREM 139. *A necessary and sufficient condition that a function $f(x)$ of $A(\alpha, a)$ should be its own cosine transform is that it should be of the form* (9.1.7), *where $\psi(s)$ is regular, and satisfies* (9.1.6), *in the strip*

$$a < \sigma < 1-a; \tag{9.7.1}$$

$$\psi(s) = O(e^{(\frac{1}{4}\pi-\alpha+\eta)|t|}) \tag{9.7.2}$$

for every positive η and uniformly in any strip interior to (9.7.1); *and c is any value of σ in* (9.7.1).

(i) *The condition is necessary.* The integral

$$\int\limits_{0}^{\infty} f(x)x^{s-1}\,dx \tag{9.7.3}$$

is absolutely convergent for $a < \sigma < 1-a$, so that $\mathfrak{F}(s)$ is regular in (9.7.1). Also, $f(x)$ belongs to L^2, and it follows from Theorem 136 that $\mathfrak{F}(s)$ satisfies (9.1.5) on $\sigma = \frac{1}{2}$, and therefore throughout (9.7.1), or, what is the same thing, $\psi(s)$ satisfies (9.1.6).

Also, $f(x)$ satisfies the conditions of Theorem 31, with $\beta = \alpha$ and $b = 1-a$. Since

$$\psi(s) = \mathfrak{F}(s)2^{-\frac{1}{2}s}/\Gamma(\tfrac{1}{2}s), \qquad |\Gamma(\tfrac{1}{2}s)| \sim Ce^{-\frac{1}{4}\pi|t|}|\tfrac{1}{2}t|^{\frac{1}{2}\sigma-\frac{1}{2}},$$

it follows that $\psi(s)$ satisfies the conditions stated.

(ii) The condition is also sufficient because Theorems 31 and 136, on which we have based the argument, are reversible.

9.8. More general conditions. The next theorem is of a more general kind here $f(x)$ does not necessarily belong to any L- or $A(\alpha, a)$-class

THEOREM 140. *Let $f(x)$ be integrable over any finite interval; let*

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\to\infty} f(y)\cos xy\, dy \tag{9.8.1}$$

exist for every x; and let

$$\mathfrak{F}(s) = \int_{\to 0}^{\to\infty} f(x)x^{s-1}\, dx \tag{9.8.2}$$

exist for $|\sigma-\frac{1}{2}| < \alpha$, where $\alpha > 0$. Then a necessary and sufficient condition that $F_c(x) = f(x)$ almost everywhere is that $\mathfrak{F}(s)$ should satisfy (9.1.5) *for $|\sigma-\frac{1}{2}| < \alpha$.*

Let $\frac{1}{2} < \beta < \frac{1}{2}+\alpha$, and

$$g(x) = \int_1^x f(\xi)\xi^{\beta-1}\, d\xi.$$

Then $g(x)$ is bounded. Hence if $\sigma < \beta$

$$\begin{aligned}\int_1^X f(x)x^{s-1}\, dx &= \int_1^X g'(x)x^{s-\beta}\, dx \\ &= g(X)X^{s-\beta}-(s-\beta)\int_1^X g(x)x^{s-\beta-1}\, dx \\ &= O(1)+O\left(|s|\int_1^X x^{\sigma-\beta-1}\, dx\right) = O(|t|)\end{aligned}$$

for all X. Similarly,

$$\int_{1/X}^1 f(x)x^{s-1}\, dx = O(|t|)$$

for $\sigma > \frac{1}{2}-\alpha$. Thus

$$\int_{1/X}^X f(x)x^{s-1}\, dx = O(|t|)$$

in any strip interior to $|\sigma-\frac{1}{2}| < \alpha$.

It follows by dominated convergence that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)\frac{x^{3-s}}{(s-1)(s-2)(s-3)}\, ds \\ &\qquad = \frac{1}{2\pi i}\int_{\to 0}^{\to\infty} f(\xi)\frac{x^3}{\xi}\, d\xi \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(x/\xi)^{-s}}{(s-1)(s-2)(s-3)}\, ds \\ &\qquad = -\tfrac{1}{2}\int_0^x f(\xi)(x-\xi)^2\, d\xi,\end{aligned}$$

and that

$$\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(s)\cos\frac{1}{2}s\pi}{(s-1)(s-2)(s-3)}\mathfrak{F}(1-s)x^{3-s}\,ds$$

$$=\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s-3)\cos\tfrac{1}{2}s\pi\;\mathfrak{F}(1-s)x^{3-s}\,ds$$

$$=\frac{x^3}{2\pi i}\int\limits_{\to 0}^{\to\infty}f(\xi)\,d\xi\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s-3)\cos\tfrac{1}{2}s\pi\,(x\xi)^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int\limits_{\to 0}^{\to\infty}\frac{f(\xi)}{\xi^3}\,d\xi\int\limits_{-2\frac{1}{2}-i\infty}^{-2\frac{1}{2}+i\infty}\Gamma(s)\sin\tfrac{1}{2}s\pi\,(x\xi)^{-s}\,ds$$

$$=\int\limits_{0}^{\to\infty}f(\xi)\frac{\sin x\xi-x\xi}{\xi^3}\,d\xi.$$

If $\mathfrak{F}(s)$ satisfies (9.1.5), it follows that

$$\frac{1}{2}\int\limits_0^x f(\xi)(x-\xi)^2\,d\xi=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^{\to\infty}f(\xi)\frac{x\xi-\sin x\xi}{\xi^3}\,d\xi.$$

But, as in the proof of Theorem 118, (9.8.1) gives

$$\int\limits_0^x du\int\limits_0^u F_c(v)\,dv=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^{\to\infty}f(\xi)\frac{1-\cos x\xi}{\xi^2}\,d\xi.$$

We may integrate over $(0,x)$ by uniform convergence; hence

$$\tfrac{1}{2}\int\limits_0^x f(\xi)(x-\xi)^2\,d\xi=\int\limits_0^x d\xi\int\limits_0^\xi du\int\limits_0^u F_c(v)\,dv,$$

and, differentiating three times, it follows that $f(x)=F_c(x)$ almost everywhere.

Conversely, if $f(x)=F_c(x)$ almost everywhere, the argument shows that

$$\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\mathfrak{F}(s)-\sqrt{(2/\pi)}\Gamma(s)\cos\frac{1}{2}s\pi\,\mathfrak{F}(1-s)}{(s-1)(s-2)(s-3)}x^{-s}\,ds=0$$

for all values of x. Since the integrand belongs to L, it must be null (Theorem 32, second part). Hence $\mathfrak{F}(s)$ satisfies (9.1.5) on $\sigma=\frac{1}{2}$, and so throughout its region of regularity.

If the conditions are satisfied, $f(x)$ is represented by (9.1.7) in the $(C, 1)$ sense, by Theorem 32.

9.9. A general theorem. Even if (as in the case $f(x) = x^{-\frac{1}{2}}$) the integral (9.8.2) does not exist for any s, it is possible to obtain a result corresponding to (9.1.5), but involving the functions $\mathfrak{F}_+(s)$, $\mathfrak{F}_-(s)$. We shall deduce our result from the following theorem, which is frequently useful.

THEOREM 141. *Let $\phi(w)$ be regular in the strip $a_1 \leqslant v \leqslant a_2$, and let $\phi(u+iv)$ be $L(-\infty,\infty)$ (or $L^2(-\infty,\infty)$), and tend to 0 as $u \to \pm\infty$, for v in the above interval. Let $\psi(w)$ have similar properties in $b_1 \leqslant v \leqslant b_2$, where $b_2 < a_1$. Let*

$$\int\limits_{ia-\infty}^{ia+\infty} \phi(w)e^{-ixw}\,dw + \int\limits_{ib-\infty}^{ib+\infty} \psi(w)e^{-ixw}\,dw = 0 \qquad (9.9.1)$$

for all x, where $a_1 < a < a_2$, $b_1 < b < b_2$. Then ϕ and ψ are regular for $b_1 < v < a_2$, their sum is 0 in this strip, and they tend to 0, as $u \to \pm\infty$, uniformly in any interior strip.

Consider first the L case. Multiply (9.9.1) by $e^{ix\zeta}$, where $\zeta = \xi + i\eta$, $a < \eta < a_2$, and integrate with respect to x over $(0, \infty)$. We can invert the order of integration by absolute convergence, and we obtain

$$\int\limits_{ia-\infty}^{ia+\infty} \frac{\phi(w)}{w-\zeta}\,dw + \int\limits_{ib-\infty}^{ib+\infty} \frac{\psi(w)}{w-\zeta}\,dw = 0 \quad (a < \eta < a_2). \qquad (9.9.2)$$

Now move the line of integration of the ϕ-integral to $v = a_2$. We obtain

$$\int\limits_{ia_2-\infty}^{ia_2+\infty} \frac{\phi(w)}{w-\zeta}\,dw + \int\limits_{ib-\infty}^{ib+\infty} \frac{\psi(w)}{w-\zeta}\,dw = -2\pi i\phi(\zeta) \quad (a < \eta < a_2). \qquad (9.9.3)$$

The left-hand side is now regular for $b < \eta < a_2$. It therefore provides the analytic continuation of $-2\pi i\phi(\zeta)$ throughout this strip.

Similarly, multiplying (9.9.1) by $e^{ix\zeta}$, where $b_1 < \eta < b$, and integrating over $(-\infty, 0)$, we obtain (9.9.2) with $b_1 < \eta < b$. Moving the line of integration of the ψ-integral to $\eta = b_1$, we obtain

$$\int\limits_{ia-\infty}^{ia+\infty} \frac{\phi(w)}{w-\zeta}\,dw + \int\limits_{ib_1-\infty}^{ib_1+\infty} \frac{\psi(w)}{w-\zeta}\,dw = 2\pi i\psi(\zeta) \quad (b_1 < \eta < b). \qquad (9.9.4)$$

This provides the analytic continuation of $2\pi i\psi(\zeta)$ over $b_1 < \eta < a$.

If $b < \eta < a$, the left-hand sides of (9.9.3) and (9.9.4) are equal, by an obvious application of Cauchy's theorem. Hence

$$\phi(\zeta) = -\psi(\zeta)$$

in the strip.

Also

$$\left|\int\limits_{ia_2-\infty}^{ia_2+\infty} \frac{\phi(w)}{w-\zeta}\,dw\right|$$

$$\leqslant \frac{1}{|a_2-\eta|}\left(\int\limits_{-\infty}^{-U} + \int\limits_{U}^{\infty}\right)|\phi(u+ia_2)|\,du + \frac{1}{|U-\xi|}\int\limits_{-U}^{U} |\phi(u+ia_2)|\,du,$$

which tends to 0 as $\xi \to \pm\infty$, by choosing first U and then ξ. Similarly, for the other term on the left of (9.9.3). Hence $\phi(\zeta) \to 0$ uniformly in the strip.

In the L^2 case (9.9.2) follows from (9.9.1) by the L^2 case of Parseval's formula, and the argument then proceeds as before; in the last part we put

$$\left|\int\limits_{ia_2-\infty}^{ia_2+\infty} \frac{\phi(w)}{w-\zeta}\,dw\right| \leqslant \left\{\int\limits_{-\infty}^{-U} \frac{du}{|w-\zeta|^2}\int\limits_{-\infty}^{-U} |\phi(u+ia_2)|^2\,du\right\}^{\frac{1}{2}} +$$

$$+\left\{\int\limits_{U}^{\infty} \frac{du}{|w-\zeta|^2}\int\limits_{U}^{\infty} |\phi(u+ia_2)|^2\,du\right\}^{\frac{1}{2}} +$$

$$+\frac{1}{|U-\xi|}\int\limits_{-U}^{U} |\phi(u+ia_2)|\,du,$$

and again choose first U and then ξ.

9.10. Application. THEOREM 142. *Let $f(x)$ be integrable over every finite interval, and tend to 0 at infinity, and let*

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{0}^{\to\infty} f(y)\cos xy\,dy \tag{9.10.1}$$

for all but a finite set of values of x. Then almost everywhere

$$f(x) = \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \mathfrak{F}_+(s)x^{-s}\,ds + \frac{1}{2\pi i}\int\limits_{-\beta-i\infty}^{\beta+i\infty} \mathfrak{F}_-(s)x^{-s}\,ds$$

$$(\beta < 0,\ \alpha > 1), \tag{9.10.2}$$

where the integrals are $(C, 1)$ *integrals;* $\mathfrak{F}_-(s)$ *is regular for* $\sigma < 0$, *and* $\mathfrak{F}_+(s)$ *for* $\sigma > 1$; *and the functions*

$$\begin{gathered}\mathfrak{F}_+(s)-\sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_-(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi,\\ \mathfrak{F}_-(s)-\sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_+(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\end{gathered} \tag{9.10.3}$$

are regular for $0 \leqslant \sigma \leqslant 1$, *except possibly for simple poles at* $s = 0$; *and their sum is* 0 *in the strip.*

Let $$f_1(x) = \int_0^x f(u)\,du, \qquad f_2(x) = \int_0^x f_1(u)\,du,$$

etc. Then, as in the proof of Theorem 113, (9.10.1) gives

$$f_2(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\frac{1-\cos xy}{y^2}\,dy. \tag{9.10.4}$$

Let $$\mathfrak{F}_-(s) = \int_1^\infty f(x)x^{s-1}\,dx, \qquad \mathfrak{F}_+(s) = \int_0^1 f(x)x^{s-1}\,dx.$$

These are clearly regular for $\sigma < 0$ and $\sigma > 1$ respectively; and (9.10.2) holds by the $(C, 1)$ analogue of Theorem 24, for Mellin integrals. Let

$$\begin{aligned}\Phi(x) = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\mathfrak{F}_-(s)\frac{x^{2-s}}{(s-1)(s-2)}\,ds\,-&\\ -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_+(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\frac{x^{2-s}}{(s-1)(s-2)}\,ds\,+&\\ +\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\mathfrak{F}_+(s)\frac{x^{2-s}}{(s-1)(s-2)}\,ds\,-&\\ -\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_-(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\frac{x^{2-s}}{(s-1)(s-2)}\,ds&\\ = \Phi_1(x)+\Phi_2(x)+\Phi_3(x)+\Phi_4(x).&\end{aligned}$$

We may insert the above integrals for $\mathfrak{F}_-(s)$, etc., and invert, by absolute convergence. We obtain

$$\begin{aligned}\Phi_1(x) &= \frac{x}{2\pi i}\int_1^\infty f(y)\,dy\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{(x/y)^{1-s}}{(s-1)(s-2)}\,ds\\ &= \int_1^x f(y)(x-y)\,dy \quad (x > 1), \qquad 0 \quad (x < 1),\end{aligned}$$

$$\Phi_3(x) = \frac{x}{2\pi i}\int\limits_0^1 f(y)\,dy \int\limits_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{(x/y)^{1-s}}{(s-1)(s-2)}\,ds$$

$$= \begin{cases} -\displaystyle\int\limits_0^1 yf(y)\,dy & (x > 1), \\ -\displaystyle\int\limits_0^x yf(y)\,dy - x\int\limits_x^1 f(y)\,dy & (x < 1). \end{cases}$$

Hence $$\Phi_1(x)+\Phi_3(x) = \int\limits_0^x f(y)(x-y)\,dy - x\int\limits_0^1 f(y)\,dy$$

for all $x > 0$. Also

$$\Phi_2(x) = -\sqrt{\left(\frac{2}{\pi}\right)}\frac{x^2}{2\pi i}\int\limits_0^1 f(y)\,dy \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s-2)\cos\tfrac{1}{2}s\pi\,(xy)^{-s}\,ds$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^1 f(y)\frac{\cos xy - 1 - \frac{1}{2}x^2y^2}{y^2}\,dy,$$

$$\Phi_4(x) = -\sqrt{\left(\frac{2}{\pi}\right)}\frac{x^2}{2\pi i}\int\limits_1^\infty f(y)\,dy \int\limits_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(s-2)\cos\tfrac{1}{2}s\pi\,(xy)^{-s}\,ds$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_1^\infty f(y)\frac{\cos xy - 1}{y^2}\,dy.$$

Altogether

$$\Phi(x) = f_2(x) - \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty f(y)\frac{1-\cos xy}{y^2}\,dy + ax + bx^2,$$

where a and b are constants. Hence, by (9.10.4),

$$\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left\{\mathfrak{F}_-(s) - \sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_+(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi + a + 2b\frac{s-1}{s}\right\}\frac{x^{2-s}\,ds}{(s-1)(s-2)} +$$

$$+ \int\limits_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \left\{\mathfrak{F}_+(s) - \sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_-(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi - a - 2b\frac{s-1}{s}\right\}\frac{x^{2-s}\,ds}{(s-1)(s-2)}$$

$$= 0$$

for every positive x. The result therefore follows from Theorem 141, with an obvious change of variable.

A similar result holds if $f(x)$ does not tend to 0, but (9.10.1) holds everywhere. The proof is similar, but with an extra factor $s-3$ in the denominator.

9.11. The Second Solution. A similar set of theorems might be constructed for the second solution obtained in § 9.2. It will perhaps be sufficient to prove one of them, and we take the case of analytic functions. We shall say that $f(x)$ belongs to $A^*(\omega, a)$, where $0 < \omega \leqslant \frac{1}{4}\pi$, $0 < a < \frac{1}{2}$, if (i) it is an analytic function of $x = re^{i\theta}$ regular in the angle A^* defined by $r > 0$, $|\theta| < \omega$, and (ii) it is $O(|x|^{-a-\frac{1}{2}-\delta})$ for small x, and $O(|x|^{a-\frac{1}{2}+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqslant \omega-\eta < \omega$.

THEOREM 143. *A necessary and sufficient condition that a function $f(x)$ of $A^*(\omega, a)$ should be its own cosine transform is that it should be of the form* (9.2.4), *where c is any positive number, the integral is the limit of an integral over* $(c-iT,\ c+iT)$, *and $\mu(s)$ has the properties*

(i) *$\mu(s) = \mu(\rho e^{i\phi})$ is an analytic function of s, regular in the angle $B(\omega, a)$ defined by $\rho > 0$, $|\phi| < \frac{1}{2}\pi+2\omega$;*

(ii) *$\mu(s)$ is $O(|s|^{-\frac{1}{2}a-\delta})$ for small s, and $O(|s|^{\frac{1}{2}a+\delta})$ for large s, for every positive δ and uniformly in any angle $|\phi| \leqslant \frac{1}{2}\pi+2\omega-\zeta < \frac{1}{2}\pi+2\omega$;*

(iii) *$\mu(s)$ satisfies* (9.2.3) *in $B(\omega, a)$.*

The conditions of regularity and order follow from Theorem 31.

It is then only a question of proving that (9.2.3) is necessary and sufficient for $f(x)$ to be self-reciprocal.

Integrating (9.2.4), we get

$$f_1(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mu(s)s^{-\frac{1}{4}}\frac{e^{\frac{1}{2}x^2s}-1}{s}\,ds$$

$$= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mu(s)s^{-\frac{5}{4}}e^{\frac{1}{2}x^2s}\,ds,$$

the other term being zero, by an obvious application of Cauchy's theorem. Again,

$$\frac{\sin xy}{y} = \frac{\sqrt{\pi}}{4\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{x^2s-y^2/(4s)}s^{-\frac{3}{2}}\,ds,$$

and hence

$$\int\limits_0^\infty f(y)\frac{\sin xy}{y}\,dy = \frac{\sqrt{\pi}}{4\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{x^2s}s^{-\frac{3}{2}}\,ds\int\limits_0^\infty f(y)e^{-y^2/(4s)}\,dy$$

$$=\frac{\sqrt{\pi}}{4\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{x^2 s}s^{-\frac{1}{2}}\chi\{(2s)^{-\frac{1}{2}}\}\,ds$$

$$=\frac{\sqrt{(2\pi)}}{4\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}s^{-\frac{1}{2}}\mu\left(\frac{1}{s}\right)\,ds,$$

the inversion being justified by absolute convergence. It follows that

$$f_1(x)-\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty f(y)\frac{\sin xy}{y}\,dy=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}s^{-\frac{1}{2}}\left\{\mu(s)-\mu\left(\frac{1}{s}\right)\right\}\,ds,$$

and hence that the condition (9.2.3) is both necessary and sufficient.

9.12. Examples.

(1) If $\psi(s)=1$ in (9.1.7), then

$$f(x)=2e^{-\frac{1}{2}x^2},\qquad f(x)=2^{\frac{1}{2}-\frac{1}{2}\nu}x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$$

in the cosine and general cases respectively. The conditions of Theorem 139 (and *a fortiori* those of the less special theorems) are satisfied.

If $\psi(s)=P(\frac{1}{2}-s)$, where $P(u)$ is an even polynomial, or an even integral function of order less than 1, we find that

$$f(x)=2\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}x^2)^n}{n!}P(2n+\tfrac{1}{2})$$

is its own cosine transform. If $P(u)$ is a polynomial, $f(x)=e^{-\frac{1}{2}x^2}Q(x^2)$, where $Q(u)$ is a polynomial.

(2) Sonine's polynomials $T_\nu^n(x)$ are defined by

$$T_\nu^n(x)=\sum_{r=0}^{n}\frac{(-1)^r x^{n-r}}{r!\,(n-r)!\,\Gamma(n+\nu-r+1)}.$$

If $$f(x)=x^\nu T_\nu^n(x)e^{-\frac{1}{2}x},$$

then

$$\phi(s)=\int\limits_0^\infty f(x)e^{-sx}\,dx=\sum_{r=0}^{n}\frac{(-1)^r}{r!\,(n-r)!\,\Gamma(n+\nu-r+1)}\int\limits_0^\infty x^{n-r+\nu}e^{-(s+\frac{1}{2})x}\,dx$$

$$=\sum_{r=0}^{n}\frac{(-1)^r}{r!\,(n-r)!}(s+\tfrac{1}{2})^{r-n-\nu-1}=\frac{1}{n!}\frac{(\frac{1}{2}-s)^n}{(\frac{1}{2}+s)^{n+\nu+1}}.$$

If $$g(x)=x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}T_\nu^n(x^2),$$

then
$$\mu(s) = s^{\frac{1}{2}\nu+\frac{1}{4}} \int\limits_0^\infty g(x)x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2s}\,dx$$
$$= \tfrac{1}{2}s^{\frac{1}{2}\nu+\frac{1}{4}} \int\limits_0^\infty f(\xi)e^{-\frac{1}{2}\xi s}\,d\xi = \frac{2^\nu}{n!}(s^{\frac{1}{2}}+s^{-\frac{1}{2}})^{-\nu-1}\left(\frac{1-s}{1+s}\right)^n.$$

Hence
$$\mu(s) = (-1)^n\mu\left(\frac{1}{s}\right),$$
and $g(x)$ is $\pm R_\nu$ according as n is even or odd.†

The parabolic cylinder functions $D_n(x)$ may be defined for integral n by
$$T^n_{-\frac{1}{2}}(x^2) = \frac{2^n}{(2n)!\sqrt{\pi}}e^{\frac{1}{2}x^2}D_{2n}(x\sqrt{2})$$
and
$$xT^n_{\frac{1}{2}}(x^2) = \frac{2^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}}e^{\frac{1}{2}x^2}D_{2n+1}(x\sqrt{2}).$$
Thus $D_{2n}(x\sqrt{2})$ is $\pm R_c$ according as n is even or odd, and $D_{2n+1}(x\sqrt{2})$ is $\pm R_s$ according as n is even or odd. This is equivalent to the self-reciprocal property of Hermite polynomials (§ 3.8). In fact it is easily verified that
$$H_{2n}(x) = (2n)!\sqrt{\pi}\,T^n_{-\frac{1}{2}}(x^2), \qquad H_{2n+1}(x) = (2n+1)!\sqrt{\pi}\,xT^n_{\frac{1}{2}}(x^2).$$

In the case $\nu = \frac{1}{2}$ Parseval's formula gives
$$\int\limits_0^\infty \{f(x)\}^2e^{-sx}\,dx = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\phi(w)\phi(s-w)\,dw$$
$$= \frac{1}{2\pi i(n!)^2}\int\limits_{c-i\infty}^{c+i\infty}\frac{(\frac{1}{2}-w)^n(\frac{1}{2}-s+w)^n}{(\frac{1}{2}+w)^{n+\frac{3}{2}}(\frac{1}{2}+s-w)^{n+\frac{3}{2}}}\,dw.$$
Denoting this by $\omega(s)$, and putting $w = w'+\frac{1}{2}s$, we obtain
$$\omega(s) = \frac{1}{2\pi i(n!)^2}\int\limits_{c'-i\infty}^{c'+i\infty}\frac{\{(\frac{1}{2}-\frac{1}{2}s)^2-w'^2\}^n}{\{(\frac{1}{2}+\frac{1}{2}s)^2-w'^2\}^{n+\frac{3}{2}}}\,dw'$$
Changing s into $1/s$, and then putting $w' = w''/s$, it follows that
$$\omega(1/s) = s^2\omega(s).$$

If now
$$\mu_1(s) = s\int\limits_0^\infty x^{-\frac{1}{2}}D^2_{2n+1}(x).x^{\frac{3}{2}}e^{-\frac{1}{2}x^2s}\,dx,$$
we have
$$\mu_1(s) = \frac{\{(2n+1)!\}^2\pi}{2^{2n+1}}s\omega(s),$$

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† A. Milne (1), B. M. Wilson (1).

and hence $\mu_1(s)$ satisfies (9.2.3). Hence†

$$x^{-\frac{1}{2}}D_{2n+1}^2(x)$$

is R_1.

For negative (not necessarily integral) n we have

$$D_n(x) = \frac{e^{-\frac{1}{4}x^2}}{\Gamma(-n)} \int\limits_0^\infty e^{-tx-\frac{1}{2}t^2}t^{-n-1}\,dt.$$

It follows that

$$\int\limits_0^\infty x^{s-1}e^{\frac{1}{4}x^2}D_n(x)\,dx = \frac{\Gamma(s)\Gamma(-\frac{1}{2}n-\frac{1}{2}s)}{2^{\frac{1}{2}n+\frac{1}{2}s+1}\Gamma(-n)}.$$

It is then easily verified from the formulae of § 9.1 that‡

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x), \qquad x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x)$$

are R_ν.

(3) If $f(x) = \operatorname{sech} x\sqrt{(\frac{1}{2}\pi)}$, we find that

$$\mathfrak{F}(s) = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}s}\Gamma(s)L(s),$$

where
$$L(s) = \frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\dots, \tag{9.12.1}$$

and $\mathfrak{F}(s)$ satisfies (9.1.5) by the functional equation for $L(s)$ (§ 2.11). This is another example of Theorem 139.

If
$$f(x) = \frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}$$

we find that $\mathfrak{F}(s) = (2\pi)^{-\frac{1}{2}s}\Gamma(s)\zeta(s)$. Taking $\nu = \frac{1}{2}$ in the formulae at the end of § 9.1, we obtain

$$\psi(s) = \tfrac{1}{2}\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \frac{\xi(s)}{s(s-1)},$$

where $\xi(s)$ is Riemann's ξ-function. This is an example of the analogue of Theorem 139 for sine transforms.

Other self-reciprocal functions are associated in a similar way with the functional equations of other Dirichlet's L-functions. For example the functions

$$\frac{\cosh(\frac{1}{2}x\sqrt{\pi})}{\cosh(x\sqrt{\pi})}, \qquad \frac{1}{1+2\cosh\{x\sqrt{(\frac{2}{3}\pi)}\}}$$

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† Mitra (1), Watson (4).

‡ Varma (1).

are R_c; they are associated with

$$L_1(s) = \sum_{r=0}^{\infty} (-1)^r \left\{\frac{1}{(4r+1)^s} + \frac{1}{(4r+3)^s}\right\},$$

$$L_2(s) = \sum_{r=0}^{\infty} \left\{\frac{1}{(3r+1)^s} - \frac{1}{(3r+2)^s}\right\}$$

respectively. And

$$\frac{\sinh(\frac{1}{2}x\sqrt{\pi})}{\cosh(x\sqrt{\pi})}, \qquad \frac{\sinh\{x\sqrt{(\frac{1}{6}\pi)}\}}{2\cosh\{x\sqrt{(\frac{2}{3}\pi)}\}-1}$$

are R_s; they are associated with

$$L_3(s) = \sum_{r=0}^{\infty} (-1)^r \left\{\frac{1}{(4r+1)^s} - \frac{1}{(4r+3)^s}\right\},$$

$$L_4(s) = \sum_{r=0}^{\infty} (-1)^r \left\{\frac{1}{(6r+1)^s} - \frac{1}{(6r+5)^s}\right\}$$

respectively.

(4) It is easily verified from (7.1.8), (7.1.9) that

$$f(x) = \cos(\tfrac{1}{2}x^2 - \tfrac{1}{8}\pi)$$

is its own cosine transform. This does not belong to any L-class, but is an example of Theorem 140. The integral (9.8.2) exists for $0 < \sigma < 2$, and $\mathfrak{F}(s) = 2^{\frac{1}{2}s-1}\Gamma(\frac{1}{2}s)\cos\frac{1}{4}\pi(s-\frac{1}{2})$ satisfies (9.1.5).

(5) The function $f(x) = x^{-\frac{1}{2}}$ is its own cosine transform, and is an example of Theorem 142. Here

$$\mathfrak{F}_+(s) = \frac{1}{s-\frac{1}{2}}, \qquad \mathfrak{F}_-(s) = -\frac{1}{s-\frac{1}{2}},$$

and

$$\mathfrak{F}_+(s) - \sqrt{\left(\frac{2}{\pi}\right)}\mathfrak{F}_-(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi = \frac{1-\sqrt{(2/\pi)}\Gamma(s)\cos\frac{1}{2}s\pi}{s-\frac{1}{2}},$$

which has a simple pole at $s = 0$, and is regular for $\sigma > 0$.

A more general example of the same kind is

$$f(x) = 2^{\frac{1}{2}a}\Gamma(\tfrac{1}{2}a)x^{-a} + 2^{\frac{1}{2}-\frac{1}{2}a}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}a)x^{a-1} \quad (0 < a < 1).$$

(6) It follows from (7.5.6) and (7.5.7) that

$$\frac{\cos\frac{1}{2}x^2 + \sin\frac{1}{2}x^2}{\cosh\{x\sqrt{(\frac{1}{2}\pi)}\}}$$

is R_c, and from (7.5.10) that

$$\frac{\sin\frac{1}{2}x^2}{\sinh\{x\sqrt{(\frac{1}{2}\pi)}\}}$$

is R_s. These are examples of Theorems 139 and 143, but $\psi(s)$ and $\mu(s)$ do not seem to be particularly simple.

(7) Taking $$f(x) = x^{\frac{1}{2}}J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2),$$
we find $$\psi(s) = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{2}\nu-\frac{3}{4}}}{\Gamma(\frac{1}{4}\nu+\frac{5}{8}+\frac{1}{4}s)\Gamma(\frac{1}{4}\nu+\frac{7}{8}-\frac{1}{4}s)},$$
so that $\psi(s) = \psi(1-s)$. By Theorem 140 $f(x)$ is its own Hankel transform of order ν. In this case, however, $f(x)$ behaves like $x^{\nu+\frac{1}{2}}$ for small x, and $x^{-\frac{1}{2}}$ for large x, and does not belong to L^2, or to any L^p for which $p \leqslant 2$.

In the case $\nu = -\frac{1}{2}$ the resulting formula is
$$\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty \sqrt{y}\,J_{-\frac{1}{4}}(\tfrac{1}{2}y^2)\cos xy\,dy = \sqrt{x}\,J_{-\frac{1}{4}}(\tfrac{1}{2}x^2).$$
Differentiating twice with respect to x, we obtain formally
$$\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty y^{\frac{5}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}y^2)\cos xy\,dy = x^{\frac{5}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}x^2).$$
This is true if the integral is taken in the $(C, 1)$ sense, but it does not come under any of our general theorems. A discussion of functions self-reciprocal in this sense is given by Mehrotra (8).

(8) Let $$f(x) = x^{\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{4}(\nu-1)}J_{\frac{1}{2}\nu-\frac{1}{2}}\{b\sqrt{(x^2-b^2)}\} \quad (x > b > 0),$$
$$0 \quad (0 < x < b).$$
Then
$$\mathfrak{F}(s) = \int\limits_b^\infty x^{s-\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{4}(\nu-1)}J_{\frac{1}{2}\nu-\frac{1}{2}}\{b\sqrt{(x^2-b^2)}\}$$
$$= b^{s-\frac{1}{2}\nu}\int\limits_0^\infty (1+u^2)^{\frac{1}{2}(s-\nu-\frac{3}{2})}u^{\frac{1}{2}\nu+\frac{1}{2}}J_{\frac{1}{2}\nu-\frac{1}{2}}(b^2u)\,du$$
$$= \frac{b^{\frac{1}{2}\nu-\frac{1}{2}}K_{\frac{1}{2}(\frac{1}{2}-s)}(b^2)}{2^{\frac{1}{2}\nu-\frac{1}{2}s-\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}$$
by (7.11.6); and, $K_\mu(x)$ being an even function of μ, (9.1.8) follows.

Here $f(x)$ is $O\{(x-b)^{\frac{1}{2}\nu-\frac{1}{2}}\}$ near $x = b$, and $O(x^{-\frac{1}{2}\nu-\frac{1}{2}})$ at infinity; it belongs to L^2 if $\nu > 0$, and to L^p and L_p^* if $\nu > |1-2/p|$. If $-1 < \nu \leqslant 0$ it is a case of Theorem 140.

(9) The function
$$f(x) = x^{\nu+\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{2}}\{a\sqrt{(x^2+a^2)}\} \quad (a > 0)$$
is R_ν (see Watson § 13.47 (2)). By Watson § 13.47 (6) we find
$$\psi(s) = 2^{-\frac{1}{2}}\left(\frac{2}{a}\right)^{\frac{1}{2}\nu+\frac{1}{2}}K_{\frac{1}{2}(s-\frac{1}{2})}(a^2).$$

9.13. Lattice-point formulae. There are some interesting examples of self-reciprocal functions in the analytic theory of numbers.† Let $r(n)$ denote the number of representations of n as a sum of two squares, and let

$$\bar{P}(x) = \sum_{0\leqslant n\leqslant x}' r(n)-\pi x, \tag{9.13.1}$$

the dash implying the insertion of a factor $\frac{1}{2}$ in the last term of the sum when x is an integer. *Then*

$$f(x) = x^{-\frac{3}{2}}\left\{\bar{P}\left(\frac{x^2}{2\pi}\right)-1\right\} \tag{9.13.2}$$

belongs to R_2.

It is clear from the definition that $f(x) = O(x^{\frac{1}{2}})$ as $x \to 0$. That $\bar{P}(x) = O(x^{\frac{1}{2}})$ as $x \to \infty$ is comparatively trivial, and in fact it is known‡ that $\bar{P}(x) = O(x^{\frac{1}{3}})$. Hence $f(x) = O(x^{-\frac{5}{6}})$ as $x \to \infty$. Hence $f(x)$ is L^2, and is L^p and L_p^* if $p > \frac{6}{5}$.

We have

$$\mathfrak{F}(s) = \int\limits_0^\infty \left\{\bar{P}\left(\frac{x^2}{2\pi}\right)-1\right\}x^{s-\frac{5}{2}}\,dx = \tfrac{1}{2}(2\pi)^{\frac{1}{2}s-\frac{3}{4}}\int\limits_0^\infty \{\bar{P}(x)-1\}x^{\frac{1}{2}s-\frac{7}{4}}\,dx,$$

the integral being convergent, and $\mathfrak{F}(s)$ analytic, for $-\frac{1}{2} < \sigma < \frac{5}{6}$. The last integral is

$$\int\limits_0^\infty \left\{\sum_{1\leqslant n\leqslant x} r(n)-\pi x\right\}x^{\frac{1}{2}s-\frac{7}{4}}\,dx = -\frac{\pi}{\frac{1}{2}s+\frac{1}{4}}+\int\limits_1^\infty \left\{\sum_{1\leqslant n\leqslant x} r(n)-\pi x\right\}x^{\frac{1}{2}s-\frac{7}{4}}\,dx,$$

and this provides the analytic continuation of $\mathfrak{F}(s)$ to $\sigma < -\frac{1}{2}$, there being a simple pole at $s = -\frac{1}{2}$. If $\sigma < -\frac{1}{2}$

$$\int\limits_1^\infty (-\pi x)x^{\frac{1}{2}s-\frac{7}{4}}\,dx = \frac{\pi}{\frac{1}{2}s+\frac{1}{4}},$$

and

$$\begin{aligned}\int\limits_1^\infty \sum_{1\leqslant n\leqslant x} r(n)x^{\frac{1}{2}s-\frac{7}{4}}\,dx &= \sum_{\nu=1}^\infty \int\limits_\nu^{\nu+1} \{r(1)+\ldots+r(\nu)\}x^{\frac{1}{2}s-\frac{7}{4}}\,dx \\ &= \sum_{\nu=1}^\infty \{r(1)+\ldots+r(\nu)\}\frac{(\nu+1)^{\frac{1}{2}s-\frac{3}{4}}-\nu^{\frac{1}{2}s-\frac{3}{4}}}{\frac{1}{2}s-\frac{3}{4}}. \\ &= -\frac{1}{\frac{1}{2}s-\frac{3}{4}}\sum_{\nu=1}^\infty r(\nu)\nu^{\frac{1}{2}s-\frac{3}{4}} = \frac{Z(\frac{3}{4}-\frac{1}{2}s)}{\frac{3}{4}-\frac{1}{2}s},\end{aligned}$$

where

$$Z(s) = \sum_{n=1}^\infty \frac{r(n)}{n^s} = 4\zeta(s)L(s),$$

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† See Hardy (17), Hardy and Titchmarsh (4), 212–3.

‡ Landau, *Vorlesungen über Zahlentheorie*, 2, 204–8.

$L(s)$ being (9.12.1). Hence

$$\mathfrak{F}(s) = \tfrac{1}{2}(2\pi)^{\frac{1}{2}s-\frac{3}{4}}\frac{Z(\frac{3}{4}-\frac{1}{2}s)}{\frac{3}{4}-\frac{1}{2}s}.$$

Now
$$Z(1-s) = \pi^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}Z(s)$$

by the functional equations for $\zeta(s)$ and $L(s)$ (pp. 65–6), or independently.† Hence $\mathfrak{F}(s)$ satisfies (9.1.8) with $\nu = 2$, and so $f(x)$ is R_2.

It follows from the analogue for J_2-transforms of Theorem 136 (p. 248) that (9.1.3), with $\nu = 2$, holds in the mean-square sense. In fact it holds in the ordinary sense, i.e.

$$\frac{1}{\xi^{\frac{3}{2}}}\left\{\bar{P}\left(\frac{\xi^2}{2\pi}\right)-1\right\} = \int\limits_0^{\to\infty}\frac{1}{y^{\frac{3}{2}}}\left\{\bar{P}\left(\frac{y^2}{2\pi}\right)-1\right\}(\xi y)^{\frac{1}{2}}J_2(\xi y)\,dy \tag{9.13.3}$$

for every positive ξ. To prove this we require the analogue for J_2-transforms of Theorem 58 (p. 83). It is easy to obtain this analogue by adapting the argument of § 8.18. There we justified the inversion of

$$\int\limits_0^{\lambda} J_\nu(xu)\sqrt{(xu)}\,du\int\limits_0^{\infty} J_\nu(uy)\sqrt{(uy)}f(y)\,dy$$

by the uniform convergence of the inner integral. If $f(x)$ is L^2, the inversion is justified by the mean convergence of the inner integral, and the result is a case of Parseval's formula for Hankel transforms. Having obtained the inversion, the rest of the proof is the same as that given in § 8.18.

Putting $y = t/\xi$, $\xi = \sqrt{(2\pi x)}$, (9.13.3) gives

$$\frac{\bar{P}(x)-1}{2\pi x} = \int\limits_0^{\to\infty}\left\{\bar{P}\left(\frac{t^2}{4\pi^2 x}\right)-1\right\}\frac{J_2(t)}{t}\,dt = \int\limits_0^{\to\infty}\bar{P}\left(\frac{t^2}{4\pi^2 x}\right)\frac{J_2(t)}{t}\,dt-\tfrac{1}{2}.$$

Now

$$\int\limits_0^{2\pi\sqrt{\{(N+1)x\}}}\bar{P}\left(\frac{t^2}{4\pi^2 x}\right)\frac{J_2(t)}{t}\,dt = \sum_{n=0}^{N}\int\limits_{2\pi\sqrt{(nx)}}^{2\pi\sqrt{\{(n+1)x\}}}\left\{r(0)+\dots+r(n)-\frac{t^2}{4\pi x}\right\}\frac{J_2(t)}{t}\,dt$$

$$= \sum_{n=0}^{N}\{r(0)+\dots+r(n)\}\left\{\frac{J_1\{2\pi\sqrt{(nx)}\}}{2\pi\sqrt{(nx)}}-\frac{J_1[2\pi\sqrt{\{(n+1)x\}}]}{2\pi\sqrt{\{(n+1)x\}}}\right\}-$$

$$-\frac{1}{4\pi x}\int\limits_0^{2\pi\sqrt{\{(N+1)x\}}} t\,J_2(t)\,dt$$

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† See e.g. Mordell (2), Potter (1).

$$= \sum_{n=0}^{N} r(n)\frac{J_1\{2\pi\sqrt{(nx)}\}}{2\pi\sqrt{(nx)}} - \{r(0)+\ldots+r(N)\}\frac{J_1[2\pi\sqrt{\{(N+1)x\}}]}{2\pi\sqrt{\{(N+1)x\}}} -$$

$$-\frac{1}{4\pi x}\left\{-2\pi\sqrt{\{(N+1)x\}}J_1[2\pi\sqrt{\{(N+1)x\}}]+2\int_0^{\to\infty} J_1(t)\,dt+o(1)\right\}$$

for $N \to \infty$, x fixed. The terms involving $J_1[2\pi\sqrt{\{(N+1)x\}}]$ are

$$-\{r(0)+\ldots+r(N)-(N+1)\pi\}\frac{J_1[2\pi\sqrt{\{(N+1)x\}}]}{2\pi\sqrt{\{(N+1)x\}}} = O(N^{-\frac{1}{4}}),$$

and, since $\int\limits_0^{\to\infty} J_1(t)\,dt = 1$, we obtain finally†

$$\bar{P}(x) = \sqrt{x}\sum_{n=1}^{\infty}\frac{r(n)}{\sqrt{n}}J_1\{2\pi\sqrt{(nx)}\}. \tag{9.13.4}$$

It has been proved by Walfisz‡ and Oppenheim|| that, if $r_p(n)$ is the number of representations of n as a sum of p squares, and

$$\bar{P}_p(x) = \sum_{0\leqslant n\leqslant x}{}' r_p(n) - \frac{\pi^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}x^{\frac{1}{2}p},$$

then
$$\bar{P}_p(x) = x^{\frac{1}{2}p}\sum_{1}^{\infty}\frac{r_p(n)}{n^{\frac{1}{2}p}}J_{\frac{1}{2}p}\{2\pi\sqrt{(nx)}\},$$

the series being summable by Cesàro's means of sufficiently high order. It follows that

$$x^{-\frac{1}{2}-\frac{1}{2}p}\left\{\bar{P}_p\left(\frac{x^2}{2\pi}\right)-1\right\}$$

belongs to $R_{1+\frac{1}{2}p}$. If we take $p = 3$, and use Walfisz's result $\bar{P}_3(x) = O(x^{\frac{11}{16}+\epsilon})$, we find that $f(x)$ falls under the obvious extension of Theorem 136. This is not true for any larger p.

If we take $p = 1$, we find that

$$f(x) = \frac{1}{x}\left(\frac{x}{\sqrt{(2\pi)}}-\left[\frac{x}{\sqrt{(2\pi)}}\right]\right),$$

where $[u]$ is the integral part of u, belongs to $R_{\frac{1}{2}}$, as may be verified directly.

9.14. Formulae connecting different classes of self-reciprocal functions.††

The simplest such formula is given by

Rule 1. *If $f(x)$ is its own cosine (sine) transform, then*

$$g(x) = \int_0^{\infty} f(t)e^{-xt}\,dt \tag{9.14.1}$$

is its own sine (cosine) transform.

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† See Hardy and Landau (1), Hardy (15). ‡ Walfisz (1), (2). || Oppenheim (1)

†† Phillips (1), Hardy and Titchmarsh (6), Mehrotra (1), (6).

Supposing, for example, that $f(x)$ is its own cosine transform, we have

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty g(t)\sin xt\,dt = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xt\,dt\int_0^\infty f(y)e^{-ty}\,dy$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\,dy\int_0^\infty e^{-ty}\sin xt\,dt$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\frac{x}{x^2+y^2}\,dy.$$

Now $\sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{x^2+y^2}$ is the cosine transform of e^{-xy}, and $f(y)$ is its own cosine transform. Hence Parseval's theorem for cosine transforms gives

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\frac{x}{x^2+y^2}\,dy = \int_0^\infty f(y)e^{-xy}\,dy = g(x),$$

and the rule follows. The example with $f(t) = \operatorname{sech}\{t\sqrt{(\frac{1}{2}\pi)}\}$, $g(x)$ R_s, has been observed by various authors. Rule 1 is a particular case of

RULE 2. *If $f(x)$ belongs to R_μ, and*

$$k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)x^{-s}\,ds, \tag{9.14.2}$$

where

$$\chi(s) = \chi(1-s), \tag{9.14.3}$$

then

$$g(x) = \int_0^\infty f(y)k(xy)\,dy \tag{9.14.4}$$

belongs to R_ν.

A general formula for $f(x)$ of R_μ is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\psi(s)x^{-s}\,ds, \tag{9.14.5}$$

where $\psi(s) = \psi(1-s)$. By (2.1.22),

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}-\frac{1}{2}s}\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\psi(1{-}_{\nu}s)2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\times$$
$$\times\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)x^{-s}\,ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac12 s}\Gamma(\tfrac14+\tfrac12\nu+\tfrac12 s)\psi_1(s)x^{-s}\,ds,$$

where $\psi_1(s) = 2^{\frac12}\Gamma(\tfrac34+\tfrac12\mu-\tfrac12 s)\Gamma(\tfrac14+\tfrac12\mu+\tfrac12 s)\psi(1-s)\chi(s).$

Since $$\psi_1(s) = \psi_1(1-s),$$

$g(x)$ is of the same form as (9.14.5), with μ replaced by ν. This proves the rule.

Since the rule is symmetrical in μ and ν, a kernel which transforms R_μ into R_ν also effects the converse transformation.

Taking $\mu = \frac12$, $\nu = -\frac12$, or vice versa, we obtain

$$k(x) = \frac{2\sqrt{\pi}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\chi(s)x^{-s}\,ds,$$

where $\chi(s)$ satisfies (9.14 3), as the general kernel which transforms R_c into R_s (or vice versa). Rule 1 is the case $\chi(s) = 1/2\sqrt{\pi}$. Taking

$$\chi(s) = \frac{1}{2\Gamma(\frac12+\frac12 s)\Gamma(1-\frac12 s)}, \qquad \frac{1}{\Gamma(\frac12 s)\Gamma(\frac12-\frac12 s)}, \qquad \frac{\sqrt{3}}{4\pi}\Gamma(s-\tfrac13)\Gamma(\tfrac23-s),$$

we obtain

$$k(x) = J_0(x), \qquad xJ_0(x), \qquad x^{-\frac16}e^{\frac12 x}K_{\frac16}(\tfrac12 x)$$

as other kernels with the same property.

Taking $\chi(s) = 2^{\frac12\mu+\frac12\nu-\frac12}$ in the general rule, we obtain

$$k(x) = x^{\frac12\mu+\frac12\nu+\frac12}K_{\frac12\nu-\frac12\mu}(x),$$

and, in particular, $K_0(x)$ transforms R_c into itself.

Taking $$\chi(s) = \frac{2^{\frac12\nu-\frac12\mu-\frac12}}{\Gamma(\frac14+\frac12\mu+\frac12 s)\Gamma(\frac34+\frac12\mu-\frac12 s)},$$

we obtain $$k(x) = x^{\frac12\nu-\frac12\mu+\frac12}J_{\frac12\mu+\frac12\nu}(x);$$

and taking $$\chi(s) = \frac{2^{\frac12\mu-\frac12\nu-\frac12}}{\Gamma(\frac14+\frac12\nu+\frac12 s)\Gamma(\frac34+\frac12\nu-\frac12 s)},$$

we obtain $$k(x) = x^{\frac12\mu-\frac12\nu+\frac12}J_{\frac12\mu+\frac12\nu}(x).$$

Naturally any of these rules, once they have been obtained, can be verified in the same way as Rule 1.

9.15. Other rules for such transformations may be obtained by combining those already known. For example, if we iterate Rule 1, we obtain

$$g(x) = \int_0^\infty e^{-xy}\,dy \int_0^\infty f(t)e^{-yt}\,dt = \int_0^\infty \frac{f(t)}{t+x}\,dt \tag{9.15.1}$$

as a function which is R_c (R_s) if $f(x)$ is R_c (R_s). This transformation is not of the above form; but it is of the form

$$g(x) = \frac{1}{x}\int_0^\infty f(y)k\left(\frac{y}{x}\right)\,dy, \tag{9.15.2}$$

with $k(u) = 1/(1+u)$. This suggests a general rule for transformations of this type.

Rule 3. *If $f(x)$ belongs to R_μ, and*

$$k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\chi(s)x^{-s}\,ds, \tag{9.15.3}$$

where $\chi(s) = \chi(1-s)$, then (9.15.2) *belongs to R_ν.*

If $f(x)$ is given by (9.14.5), (2.1.17) gives

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi_1(s)x^{-s}\,ds,$$

where $\psi_1(s) = \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\psi(s)\chi(1-s)$.

This verifies the rule as before.

In the particular case $\mu = \nu$, (9.15.3) reduces simply to

$$k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi_1(s)x^{-s}\,ds,$$

where $\chi_1(s) = \chi_1(1-s)$; and this is equivalent to

$$k\left(\frac{1}{x}\right) = xk(x). \tag{9.15.4}$$

Hence

Rule 4. *If $f(x)$ belongs to R_ν, and $k(x)$ satisfies* (9.15.4), *then $g(x)$, defined by* (9.15.2), *belongs to R_ν.*

It is easy to verify this directly in the usual way.

Particular cases are

$$k(x) = \frac{1}{(1+x^\alpha)^{1/\alpha}},$$

or, more generally,

$$k(x) = \frac{x^{\frac{1}{2}(\alpha\beta-1)}}{(1+x^\alpha)^\beta}.$$

Taking $f(x) = x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$, which belongs to R_ν, and $\alpha = 2$, $\beta = 1-\nu$, we obtain

$$g(x) = \frac{1}{x}\int\limits_0^\infty y^{\nu+\frac{1}{2}}e^{-\frac{1}{2}y^2}\left(\frac{y}{x}\right)^{\frac{1}{2}-\nu}\left(1+\frac{y^2}{x^2}\right)^{\nu-1} dy$$

$$= x^{\frac{1}{2}-\nu}e^{\frac{1}{2}x^2}\int\limits_x^\infty t^{2\nu-1}e^{-\frac{1}{2}t^2}\, dt$$

(putting $x^2+y^2 = t^2$) as a function of R_ν. In particular†

$$e^{\frac{1}{2}x^2}\int\limits_x^\infty e^{-\frac{1}{2}t^2}\, dt$$

is R_s.

The formula $$g(x) = x^{\frac{1}{2}+\nu}\int\limits_0^\infty y^{\nu+\frac{1}{2}}h(y)e^{-\frac{1}{2}x^2y^2}\, dy$$

for functions of R_ν, where $h(y) = h(1/y)$, is derivable from the above rule by taking $f(x) = x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$, and making obvious transformations.

Taking $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$, we obtain

RULE 5. *If $f(x)$ belongs to R_c, and*

$$k(x) = \frac{1}{2i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\chi(s)}{\sin\frac{1}{2}s\pi}x^{-s}\, ds,$$

where $\chi(s) = \chi(1-s)$, then

$$g(x) = \frac{1}{x}\int\limits_0^\infty f(y)k\left(\frac{y}{x}\right) dy$$

belongs to R_s.

For example, if $\chi(s) = 1$, then $k(x) = 1/(1+x^2)$, and

$$g(x) = x\int\limits_0^\infty \frac{f(y)}{x^2+y^2}\, dy.$$

If $\chi(s) = \sqrt{\pi}/\Gamma(\frac{1}{2}+\frac{1}{2}s)\Gamma(1-\frac{1}{2}s)$, then

$$k(x) = (1-x^2)^{-\frac{1}{2}} \quad (0 < x < 1), \qquad 0 \quad (x > 1),$$

and $$g(x) = \int\limits_0^x \frac{f(y)\, dy}{\sqrt{(x^2-y^2)}}.$$

There are, of course, similar rules for transformations from R_s to R_c.

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† Hardy (1).

9.16. The main interest of the above rules is in their formal appearance. A considerable variety of theorems about them could be constructed.† We shall give here only one, applying to Rule 2. The reader should have little difficulty in dealing with the other rules in the same way.

THEOREM 144. *Let $f(x)$ and $k(x)$ be $L^2(0,\infty)$, and let $f(x)$ be R_μ. Let*

$$g(x) = \int_0^\infty f(y)k(xy)\,dy \tag{9.16.1}$$

be also $L^2(0,\infty)$. Then, in order that $g(x)$ should be R_ν, it is necessary and sufficient that $\mathfrak{K}(s)$, the Mellin transform of $k(x)$, should be of the form

$$\mathfrak{K}(s) = 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s), \tag{9.16.2}$$

where $\chi(s) = \chi(1-s)$, and the right-hand side is $L^2(\tfrac{1}{2}-i\infty, \tfrac{1}{2}+i\infty)$.

By Theorem 72, with $g(x)$ replaced by $k(xy)$.

$$\int_0^\infty f(y)k(xy)\,dy = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)\mathfrak{K}(1-s)y^{s-1}\,ds.$$

The Mellin transform of $g(x)$ is therefore

$$\mathfrak{G}(s) = \mathfrak{F}(1-s)\mathfrak{K}(s).$$

By the analogue for R_μ of Theorem 136

$$\mathfrak{F}(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\psi(s),$$

where $\psi(s) = \psi(1-s)$. If $g(x)$ is R_ν, we have also

$$\mathfrak{G}(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s),$$

where $\omega(s) = \omega(1-s)$. Hence

$$\mathfrak{K}(s) = \frac{2^{\frac{1}{2}s}\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\omega(s)}{2^{\frac{1}{2}-\frac{1}{2}s}\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)\psi(1-s)},$$

which is of the form (9.16.2), with

$$\chi(s) = \sqrt{2}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\omega(s)/\psi(1-s)$$

satisfying $\chi(s) = \chi(1-s)$. Hence the form (9.16.2) is necessary; and the reversed argument shows that it is sufficient.

9.17. A series formula for self-reciprocal functions may be derived from the function

$$f(x) = \frac{1}{x}\left(\frac{x}{\sqrt{(2\pi)}} - \left[\frac{x}{\sqrt{(2\pi)}}\right]\right),$$

---

† See e.g. Mehrotra (1).

which belongs to $R_{\frac{1}{2}}$. Let $k(x)$ be a kernel which transforms R_c to R_ν. Then, by Rule 3,

$$k(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}2^s\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)x^{-s}\,ds;$$

and
$$xk'(x)=-\frac{1}{\pi i}\int\limits_{c-i\infty}^{c+i\infty}2^s\Gamma(1+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)x^{-s}\,ds$$

is a kernel which transforms $R_{\frac{1}{2}}$ to R_ν. Hence

$$\begin{aligned}g(x)&=\int\limits_0^\infty f(y)xyk'(xy)\,dy\\&=\sum_{n=1}^\infty x\int\limits_{(n-1)\sqrt{(2\pi)}}^{n\sqrt{(2\pi)}}\left\{\frac{y}{\sqrt{(2\pi)}}-n+1\right\}k'(xy)\,dy\\&=\sum_{n=1}^\infty\left\{\left[\left(\frac{y}{\sqrt{(2\pi)}}-n+1\right)k(xy)\right]_{(n-1)\sqrt{(2\pi)}}^{n\sqrt{(2\pi)}}-\frac{1}{\sqrt{(2\pi)}}\int\limits_{(n-1)\sqrt{(2\pi)}}^{n\sqrt{(2\pi)}}k(xy)\,dy\right\}\\&=\sum_{n=1}^\infty k\{nx\sqrt{(2\pi)}\}-\frac{1}{\sqrt{(2\pi)}x}\int\limits_0^\infty k(u)\,du\end{aligned}$$

should be a function of R_ν.

The rule may be verified as follows. Let

$$k(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\mathfrak{R}(s)x^{-s}\,ds\quad(c>0).$$

Then

$$\begin{aligned}\sum_{n=1}^\infty k\{nx\sqrt{(2\pi)}\}&=\frac{1}{2\pi i}\int\limits_{c'-i\infty}^{c'+i\infty}\mathfrak{R}(s)\sum_{n=1}^\infty\{nx\sqrt{(2\pi)}\}^{-s}\,ds\quad(c'>1)\\&=\frac{1}{2\pi i}\int\limits_{c'-i\infty}^{c'+i\infty}\mathfrak{R}(s)(2\pi)^{-\frac{1}{2}s}\zeta(s)x^{-s}\,ds\\&=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\mathfrak{R}(s)(2\pi)^{-\frac{1}{2}s}\zeta(s)x^{-s}\,ds+\frac{\mathfrak{R}(1)}{\sqrt{(2\pi)}x}.\end{aligned}$$

Hence
$$g(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\mathfrak{R}(s)(2\pi)^{-\frac{1}{2}s}\zeta(s)x^{-s}\,ds.$$

If $k(x)$ transforms R_c to R_ν, then by Rule 3

$$\mathfrak{K}(s) = 2^s\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s),$$

where $\chi(s) = \chi(1-s)$. Hence

$$\begin{aligned}\mathfrak{G}(s) &= \mathfrak{K}(s)(2\pi)^{-\frac{1}{2}s}\zeta(s)\\ &= 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi_1(s),\end{aligned}$$

where $$\chi_1(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)\chi(s).$$

Hence $$\chi_1(s) = \chi_1(1-s)$$

by the functional equation for $\zeta(s)$.

Hence $g(x)$ belongs to R_ν, by (9.1.9).

As examples, let $k(x) = e^{-x}$, $\nu = \frac{1}{2}$. Then

$$g(x) = \frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}$$

is R_s, as in § 9.12 (3).

Taking $k(x) = J_0(x)$, another R_s function is†

$$\sum_{n=1}^{\infty} J_0\{nx\sqrt{(2\pi)}\}-\frac{1}{x\sqrt{(2\pi)}} = \sqrt{\left(\frac{2}{\pi}\right)} \sum_{n<x/\sqrt{(2\pi)}} \frac{1}{\sqrt{(x^2-2n^2\pi)}}-\frac{1}{2}.$$

Taking $k(x) = x^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}(x)$, we obtain‡

$$\sum_{n=1}^{\infty}\{nx\sqrt{(2\pi)}\}^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\{nx\sqrt{(2\pi)}\}-\frac{2^{\frac{1}{2}\nu-\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}\nu)}{x}$$

as a function of R_ν.

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† See § 2.10 (vi).

‡ Watson (1).

X

DIFFERENTIAL AND DIFFERENCE EQUATIONS

10.1. Introduction. In this chapter we use Fourier's integral formula and its related formulae to obtain the solutions of certain differential equations. The general method is to transform a differential (or other functional) equation, involving an unknown function, into a relation involving the Fourier transform, or some similar transform, of the original function. The new relation may be simpler, and so lead to the solution.

That certain differential equations can be solved by means of definite integrals was shown by Laplace and Cauchy. The main object of this chapter is to present some cases of this familiar method as exercises in the use of Fourier's integral formula.

The chapter is merely a collection of examples illustrating the possibilities of the method. Most of them are familiar, and the solutions are to be found in standard works. The methods usually employed, however, are more or less tentative, and often make no explicit use of Fourier's theorem. Here we aim at solving the equations subject to simple conditions which justify *a priori* the process used.

10.2. Ordinary differential equations. We shall first give a method of solving ordinary linear differential equations, due to Bromwich.[†] The method, in its rigorous form, depends on Theorems 33 and 34.

One of Bromwich's examples is

$$\left.\begin{aligned}\left(\frac{d^2}{dt^2}-4\frac{d}{dt}\right)x-\left(\frac{d}{dt}-1\right)y &= 0,\\ \left(\frac{d}{dt}+6\right)x+\left(\frac{d^2}{dt^2}-\frac{d}{dt}\right)y &= 0,\end{aligned}\right\} \tag{10.2.1}$$

where $x(0) = x_0$, $x'(0) = x_1$, $y(0) = y_0$, $y'(0) = y_1$ are given constants.

It can be seen *a priori* that $x(t)$ and $y(t)$ are integral functions of exponential type. For example, by further differentiation and elimination we obtain

$$x'''(t)+c_1x''(t)+c_2x'(t)+c_3x(t) = 0.$$

[†] Bromwich (1).

Hence $$x^{(n+3)}(t)+c_1x^{(n+2)}(t)+c_2x^{(n+1)}(t)+c_3x^{(n)}(t) = 0.$$

If $|x^{(m)}(t)| \leqslant \{K(t)\}^m$ for $m = 1, 2,..., n+2$, it follows that

$$|x^{(n+3)}(t)| \leqslant (|c_1|+...+|c_3|)\{K(t)\}^{n+2} \leqslant \{K(t)\}^{n+3}$$

provided that $K(t) \geqslant 1$, $K(t) \geqslant |c_1|+...+|c_3|$. Hence $x(t)$ is expansible by Taylor's theorem, and

$$|x(t)| = \left|\sum_{n=0}^{\infty} \frac{x^{(n)}(0)t^n}{n!}\right| \leqslant \sum_{n=0}^{\infty} \frac{\{K(0)\}^n|t|^n}{n!} = e^{K(0)|t|},$$

so that $x(t)$ is an integral function of exponential type. Similarly for $y(t)$.

Hence, by Theorem 33,

$$x(t) = \frac{1}{2\pi i}\int\limits_C \xi(w)e^{wt}\,dw, \qquad y(t) = \frac{1}{2\pi i}\int\limits_C \eta(w)e^{wt}\,dw, \tag{10.2.2}$$

where $\xi(w)$ and $\eta(w)$ are regular for $|w| > R$, say, and zero at infinity; and C is a simple closed curve surrounding $|w| = R$.

The differential equations then give

$$\left.\begin{aligned} \int\limits_C \{\xi(w)(w^2-4w)-\eta(w)(w-1)\}e^{wt}\,dw &= 0, \\ \int\limits_C \{\xi(w)(w+6)+\eta(w)(w^2-w)\}e^{wt}\,dw &= 0. \end{aligned}\right\} \tag{10.2.3}$$

Let $$\left.\begin{aligned} \xi(w)(w^2-4w)-\eta(w)(w-1) &= p(w), \\ \xi(w)(w+6)+\eta(w)(w^2-w) &= q(w). \end{aligned}\right\} \tag{10.2.4}$$

Then $p(w)$ and $q(w)$ are regular for $|w| > R$, except for poles of the first order at infinity. Hence, by Theorem 34, they are linear functions of w, say

$$p(w) = a+bw, \qquad q(w) = \alpha+\beta w. \tag{10.2.5}$$

Also, from (10.2.2),

$$x_0 = \frac{1}{2\pi i}\int\limits_C \xi(w)\,dw, \qquad x_1 = \frac{1}{2\pi i}\int\limits_C \xi(w)w\,dw,$$

and hence, by Laurent's theorem,

$$\xi(w) = \frac{x_0}{w}+\frac{x_1}{w^2}+O\left(\frac{1}{|w|^3}\right) \tag{10.2.6}$$

for large $|w|$. Similarly,

$$\eta(w) = \frac{y_0}{w}+\frac{y_1}{w^2}+O\left(\frac{1}{|w|^3}\right). \tag{10.2.7}$$

Substituting (10.2.5), (10.2.6), (10.2.7) in (10.2.4), and equating coefficients, we find

$$p(w) = (w-4)x_0+x_1-y_0, \qquad q(w) = x_0+(w-1)y_0+y_1.$$

Solving (10.2.4) for ξ and η, we obtain

$$\xi(w) = \frac{wp(w)+q(w)}{(w+1)(w-2)(w-3)}, \quad \eta(w) = \frac{-(w+6)p(w)+(w^2-4w)q(w)}{(w^2-1)(w-2)(w-3)}.$$

The values of $x(t)$ and $y(t)$ now follow from (10.2.2) by the calculus of residues. For example, the term in $x(t)$ corresponding to the pole at $w = -1$ is

$$\frac{-p(-1)+q(-1)}{12}e^{-t} = \frac{6x_0-x_1-y_0+y_1}{12}e^{-t}.$$

Naturally the method is quite general. Another simple example is

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x = 0,$$

$$\frac{d^2y}{dt^2}+2n\frac{dy}{dt}+n^2y = \mu\frac{dx}{dt},$$

where $x(0) = 0$, $x'(0) = \lambda$, $y(0) = 0$, $y'(0) = 0$. This is given by Jeffreys, *Operational Methods*, § 3.31, as an example of the operational equivalent of the above method.

10.3. If we are given a linear equation whose coefficients are polynomials of degree m, and treat it by the above method, we have to integrate by parts m times, and the transform of the original function satisfies a differential equation of the mth order. Consider, for example, Bessel's equation,

$$\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}+\left(1-\frac{\nu^2}{x^2}\right)z = 0,$$

where $\nu > 0$. Putting $z = x^\nu y$, we obtain

$$\frac{d^2y}{dx^2}+\frac{2\nu+1}{x}\frac{dy}{dx}+y = 0.$$

Let us seek a solution which is an integral function of exponential type. Let it be

$$y = \frac{1}{2\pi i}\int\limits_C \eta(w)e^{xw}\,dw.$$

Then
$$xy = -\frac{1}{2\pi i}\int\limits_C \eta'(w)e^{xw}\,dw,$$
$$\frac{dy}{dx} = \frac{1}{2\pi i}\int\limits_C \eta(w)we^{xw}\,dw,$$
$$x\frac{d^2y}{dx^2} = -\frac{1}{2\pi i}\int\limits_C \{2w\eta(w)+w^2\eta'(w)\}e^{xw}\,dw.$$
Hence
$$\int\limits_C \{(1+w^2)\eta'(w)-(2\nu-1)w\eta(w)\}e^{xw}\,dw = 0.$$

The factor in brackets has at most a pole of the first order at infinity; hence, by Theorem 34,
$$(1+w^2)\eta'(w)-(2\nu-1)w\eta(w) = a,$$
$$\eta(w) = a(1+w^2)^{\nu-\frac{1}{2}}\int \frac{dw}{(1+w^2)^{\nu+\frac{1}{2}}}+b(1+w^2)^{\nu-\frac{1}{2}}.$$
Since $\eta(w)$ is regular at infinity, $b = 0$, and
$$\eta(w) = a(1+w^2)^{\nu-\frac{1}{2}}\int\limits_w^\infty \frac{d\zeta}{(\zeta^2+1)^{\nu+\frac{1}{2}}},$$
where we take the branches of $(w^2+1)^{\nu-\frac{1}{2}}$ and $(\zeta^2+1)^{\nu+\frac{1}{2}}$ which are real on the real axis, and suppose the plane cut along the imaginary axis from $-i$ to i. Then
$$y = \frac{a}{2\pi i}\int\limits_C (1+w^2)^{\nu-\frac{1}{2}}e^{xw}\,dw\int\limits_w^\infty \frac{d\zeta}{(\zeta^2+1)^{\nu+\frac{1}{2}}}.$$

This can be reduced to a more familiar form. Let w be a point on the imaginary axis between 0 and i, w' the same point after a circuit has been made round i. Then
$$\eta(w)-\eta(w')$$
$$= a(1+w^2)^{\nu-\frac{1}{2}}\int\limits_w^\infty \frac{d\zeta}{(\zeta^2+1)^{\nu+\frac{1}{2}}}-ae^{2\pi i(\nu-\frac{1}{2})}(1+w^2)^{\nu-\frac{1}{2}}\int\limits_w^{-\infty} \frac{d\zeta}{(\zeta^2+1)^{\nu+\frac{1}{2}}}.$$
Now
$$\int\limits_w^{-\infty} \frac{d\zeta}{(\zeta^2+1)^{\nu+\frac{1}{2}}} = -e^{-2\pi i(\nu+\frac{1}{2})}\int\limits_{-\infty}^{w} \frac{d\zeta}{(\zeta^2+1)_1^{\nu+\frac{1}{2}}},$$
where the suffix denotes the branch obtained by cuts from $-i\infty$ to $-i$ and i to $i\infty$. Hence
$$\eta(w)-\eta(w') = a(1+w^2)^{\nu-\frac{1}{2}}\int\limits_{-\infty}^{\infty} \frac{d\zeta}{(\zeta^2+1)_1^{\nu+\frac{1}{2}}} = aK(1+w^2)^{\nu-\frac{1}{2}},$$

where K depends on ν only. Hence

$$y = K_1 \int\limits_{-1}^{1} (1-v^2)^{\nu-\frac{1}{2}} e^{ixv}\,dv = K_2 \int\limits_{0}^{1} (1-v^2)^{\nu-\frac{1}{2}} \cos xv\,dv.$$

10.4. If we put $z = x^{-\nu}y$, $x^2 = 4t$ in Bessel's equation, we obtain

$$t\frac{d^2y}{dt^2}+(1-\nu)\frac{dy}{dt}+y = 0. \tag{10.4.1}$$

The solution corresponding to $z = J_\nu(x)$ is not an integral function for general ν. Let us look instead for solutions $y(t)$ such that $y(t) = O(e^{ct})$ as $t \to \infty$ for some positive c, and $y(0) = 0$.

Such a solution is representable by a Fourier integral, by Theorem 24. Let $f(t) = y(t)$ for $t > 0$, and $f(t) = 0$ for $t < 0$. Then $F_-(w) = 0$, and $F_+(w)$ is

$$Y(w) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{0}^{\infty} y(t)e^{iwt}\,dt \tag{10.4.2}$$

for $v > c$. Since $y(t)$ is continuous and of bounded variation in any finite interval, Theorem 24 gives

$$y(t) = \frac{1}{\sqrt{(2\pi)}} \lim_{\lambda\to\infty} \int\limits_{ia-\lambda}^{ia+\lambda} Y(w)e^{-iwt}\,dw, \tag{10.4.3}$$

for $t > 0$, $a > c$.

Integrating (10.4.2) by parts, we have

$$\sqrt{(2\pi)}Y(w) = -\frac{1}{iw} \int\limits_{0}^{\to\infty} y'(t)e^{iwt}\,dt, \tag{10.4.4}$$

the integrated term vanishing. Similarly,

$$\sqrt{(2\pi)}Y'(w) = \int\limits_{0}^{\infty} ity(t)e^{iwt}\,dt$$

$$= -\frac{1}{w} \int\limits_{0}^{\to\infty} \{y(t)+ty'(t)\}e^{iwt}\,dt. \tag{10.4.5}$$

Integrating by parts again,

$$\sqrt{(2\pi)}Y'(w) = \frac{1}{iw^2} \int\limits_{0}^{\to\infty} \{2y'(t)+ty''(t)\}e^{iwt}\,dt$$

$$= \frac{1}{iw^2} \int\limits_{0}^{\to\infty} \{(\nu+1)y'(t)-y(t)\}e^{iwt}\,dt \tag{10.4.6}$$

by (10.4.1). The last integral is convergent since (10.4.4) is; the integrated term

$$-\frac{1}{iw^2}\{y(t)+ty'(t)\}e^{iwt}$$

must therefore tend to a limit as $t \to \infty$, and the limit can only be 0.

From (10.4.2), (10.4.4), and (10.4.6) it follows that

$$Y'(w) = \left(-\frac{\nu+1}{w}-\frac{1}{iw^2}\right)Y(w).$$

Hence

$$Y(w) = Kw^{-\nu-1}e^{1/iw},$$

$$y(t) = \frac{K}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty}\frac{e^{-iwt+1/iw}}{w^{\nu+1}}\,dw.$$

This is in fact a multiple of $t^{\frac{1}{2}\nu}J_\nu(2\sqrt{t})$, by (7.13.9).

10.5. A similar method may be used to solve differential equations with a given function on the right-hand side. To take a simple case, consider

$$\frac{d^2y}{dt^2}+k^2y = \phi(t) \quad (t \geqslant 0),$$

where all the functions concerned are of the form $O(e^{ct})$ as $t \to \infty$. If $v > c$,

$$\sqrt{(2\pi)}Y(w) = \int\limits_0^\infty y(t)e^{iwt}\,dt$$

$$= -\frac{y(0)}{iw}-\frac{y'(0)}{w^2}-\frac{1}{w^2}\int\limits_0^\infty y''(t)e^{iwt}\,dt,$$

integrating by parts twice. Hence

$$\Phi(w) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \phi(t)e^{iwt}\,dt = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \{y''(t)+k^2y(t)\}e^{iwt}\,dt$$

$$= -w^2Y(w)+\frac{iwy(0)}{\sqrt{(2\pi)}}-\frac{y'(0)}{\sqrt{(2\pi)}}+k^2Y(w),$$

i.e.

$$Y(w) = \frac{\Phi(w)}{k^2-w^2}-\frac{1}{\sqrt{(2\pi)}}\frac{iwy(0)-y'(0)}{k^2-w^2}.$$

Hence for a sufficiently large, in particular $a > c$,

$$y(t) = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda}\left\{\frac{\Phi(w)}{k^2-w^2}-\frac{1}{\sqrt{(2\pi)}}\frac{iwy(0)-y'(0)}{k^2-w^2}\right\}e^{-iwt}\,dw.$$

In the first part we can insert the Fourier integral for $\Phi(w)$, and invert, by absolute convergence. We obtain

$$\frac{1}{2\pi}\int_0^\infty \phi(x)\,dx \int_{ia-\infty}^{ia+\infty} \frac{e^{iw(x-t)}}{k^2-w^2}\,dw = \frac{1}{k}\int_0^t \phi(x)\sin k(t-x)\,dx,$$

evaluating the inner integral by the calculus of residues. The remaining terms are

$$y(0)\cos kt + y'(0)\frac{\sin kt}{k}.$$

Hence

$$y(t) = y(0)\cos kt + y'(0)\frac{\sin kt}{k} + \frac{1}{k}\int_0^t \phi(x)\sin k(t-x)\,dx,$$

the usual solution.

10.6. Partial differential equations. *Obtain the solution $v(x,t)$ of*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0) \tag{10.6.1}$$

such that $$v(x,0) = f(x) \quad (-\infty < x < \infty).$$

This is the classical problem of the flow of heat in an infinite rod with a given initial temperature distribution, v being the temperature, t the time, and x the distance along the rod.†

Formally we proceed as follows. Let

$$V(\xi,t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} v(x,t)e^{ix\xi}\,dx.$$

Then

$$\frac{\partial V}{\partial t} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{\partial v}{\partial t}e^{ix\xi}\,dx = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{\partial^2 v}{\partial x^2}e^{ix\xi}\,dx$$

$$= -\frac{\xi^2}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} ve^{ix\xi}\,dx = -\xi^2 V,$$

integrating by parts twice, and assuming that the terms at the limits vanish. Hence

$$V(\xi,t) = A(\xi)e^{-\xi^2 t},$$

where $A(\xi)$ depends on ξ only. Putting $t = 0$,

$$A(\xi) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(x)e^{ix\xi}\,dx = F(\xi),$$

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† Riemann-Weber, 2, § 36; Carslaw, *Heat*, § 16.

F being the transform of f. Hence

$$V(\xi,t) = F(\xi)e^{-\xi^2 t},$$

and the solution is

$$v(x,t) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{\infty} F(\xi)e^{-\xi^2 t-ix\xi}\, d\xi,$$

or, in terms of $f(x)$,

$$v(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-\xi^2 t-ix\xi}\, d\xi \int\limits_{-\infty}^{\infty} f(u)e^{i\xi u}\, du$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(u)\, du \int\limits_{-\infty}^{\infty} e^{-\xi^2 t-i\xi(x-u)}\, d\xi$$

$$= \frac{1}{2\sqrt{(\pi t)}} \int\limits_{-\infty}^{\infty} f(u)e^{-\frac{1}{4}(x-u)^2/t}\, du. \tag{10.6.2}$$

That in fact $v(x,t) \to f(x)$ as $t \to 0$ follows from the theory of Weierstrass's singular integral. The method would be justified e.g. if all the functions concerned belong to $L(-\infty,\infty)$. But the following procedure is much more general.

Suppose that $|v(x,t)| < Ke^{c|x|}$ *for some* c *and all* t, *with similar conditions on any of the partial derivatives which occur.* We shall say that such a function is of exponential type.

Let $v(x,t) \to f(x)$, as $t \to 0$, for almost all values of x. Then $|f(x)| < Ke^{c|x|}$ almost everywhere.

Let

$$V_+(\zeta,t) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{0}^{\infty} v(x,t)e^{ix\zeta}\, dx, \qquad V_-(\zeta,t) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{0} v(x,t)e^{ix\zeta}\, dx,$$

where $\zeta = \xi+i\eta$. Then V_+ exists and is regular for $\eta > c$, V_- for $\eta < -c$.

Now if $\eta > c$,

$$\sqrt{(2\pi)}\frac{\partial V_+}{\partial t} = \int\limits_{0}^{\infty} \frac{\partial v}{\partial t} e^{ix\zeta}\, dx = \int\limits_{0}^{\infty} \frac{\partial^2 v}{\partial x^2} e^{ix\zeta}\, dx$$

$$= \left[\frac{\partial v}{\partial x} e^{ix\zeta}\right]_0^\infty - i\zeta \int\limits_{0}^{\infty} \frac{\partial v}{\partial x} e^{ix\zeta}\, dx$$

$$= \left[\frac{\partial v}{\partial x} e^{ix\zeta}\right]_0^\infty - i\zeta[ve^{ix\zeta}]_0^\infty - \zeta^2 \int\limits_{0}^{\infty} ve^{ix\zeta}\, dx$$

$$= -v_x(0,t)+i\zeta v(0,t)-\zeta^2\sqrt{(2\pi)}V_+(\zeta,t).$$

This is an ordinary differential equation, of which (e.g. as in § 10.5) the solution is

$$V_+(\zeta,t) = A(\zeta)e^{-\zeta^2 t}-\frac{e^{-\zeta^2 t}}{\sqrt{(2\pi)}}\int\limits_0^t \{v_x(0,\tau)-i\zeta v(0,\tau)\}e^{\zeta^2\tau}\,d\tau.$$

Making $t \to 0$,

$$A(\zeta) = \lim_{t\to 0} V_+(\zeta,t) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty f(x)e^{ix\zeta}\,dx = F_+(\zeta)$$

by dominated convergence, since $v(x,t) \to f(x)$ for almost all x, and

$$|v(x,t)e^{ix\zeta}| < Ke^{(c-\eta)x}.$$

Hence $$V_+(\zeta,t) = F_+(\zeta)e^{-\zeta^2 t}-\chi(\zeta,t),$$

where $\chi(\zeta,t)$ is an integral function of ζ which $\to 0$ as $\xi \to \pm\infty$.

In the corresponding argument with V_- the integrated terms appear with the opposite sign, and we obtain

$$V_-(\zeta,t) = F_-(\zeta)e^{-\zeta^2 t}+\chi(\zeta,t).$$

Hence, by Theorem 24,

$$v(x,t) = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda}\{F_+(\zeta)e^{-\zeta^2 t}-\chi(\zeta,t)\}e^{-ix\zeta}\,d\zeta +$$
$$+\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ib-\lambda}^{ib+\lambda}\{F_-(\zeta)e^{-\zeta^2 t}+\chi(\zeta,t)\}e^{-ix\zeta}\,d\zeta.$$

The contribution of χ is plainly 0. The contribution of F_+ is

$$\frac{1}{2\pi}\int\limits_{ia-\infty}^{ia+\infty} e^{-\zeta^2 t-ix\zeta}\,d\zeta\int\limits_0^\infty f(u)e^{iu\zeta}\,du = \frac{1}{2\pi}\int\limits_0^\infty f(u)\,du\int\limits_{ia-\infty}^{ia+\infty} e^{-\zeta^2 t-i\zeta(x-u)}\,d\zeta$$

(inverting by absolute convergence)

$$= \frac{1}{2\sqrt{(\pi t)}}\int\limits_0^\infty f(u)e^{-(x-u)^2/(4t)}\,du.$$

Similarly for F_-, and we obtain (10.6.2) as before.

We do not know whether, for a unique solution, it is necessary to assume that $v(x,t)$ is of exponential type. But some condition bearing on $v(x,t)$ and not merely on $f(x)$ is necessary. It is an easily verified rule that, if $v(x,t)$ is a solution of (10.6.1), then so is

$$t^{-\frac12}e^{-\frac14 x^2/t}v\left(\frac{x}{t},-\frac{1}{t}\right).$$

Hence, $v(x,t) = x$ being a solution, so is

$$xt^{-\frac{3}{2}}e^{-\frac{1}{4}x^2/t},$$

as is easily verified; and this function tends to 0, as $t \to 0$, for every x, without vanishing identically. It is of course unbounded near $x = 0$ as $t \to 0$ (e.g. for $x = \surd t$), so that it does not satisfy the conditions of the above analysis.

10.7. *Obtain the solution* $v(x,t)$ *of*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad (x > 0,\ t > 0)$$

such that $v(x,0) = 0$ $(x > 0)$, $v(0,t) = f(t)$ $(t > 0)$.

This is the problem† of the conduction of heat in a semi-infinite rod, initially at zero temperature, the end being suddenly raised to, and maintained at, a given temperature $f(t)$.

For a formal solution let

$$V_s(\xi,t) = \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty v(x,t)\sin\xi x\,dx.$$

Then

$$\begin{aligned}\frac{\partial V_s}{\partial t} &= \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty \frac{\partial v}{\partial t}\sin\xi x\,dx \\ &= \sqrt{\left(\frac{2}{\pi}\right)} \int\limits_0^\infty \frac{\partial^2 v}{\partial x^2}\sin\xi x\,dx \\ &= -\sqrt{\left(\frac{2}{\pi}\right)}\xi \int\limits_0^\infty \frac{\partial v}{\partial x}\cos\xi x\,dx \\ &= -\sqrt{\left(\frac{2}{\pi}\right)}\xi\left\{[v\cos\xi x]_0^\infty + \xi\int\limits_0^\infty v\sin\xi x\,dx\right\} \\ &= \sqrt{\left(\frac{2}{\pi}\right)}\xi f(t) - \xi^2 V_s.\end{aligned}$$

Hence

$$V_s = A(\xi)e^{-\xi^2 t} + \sqrt{\left(\frac{2}{\pi}\right)}\xi e^{-\xi^2 t}\int\limits_0^t e^{\xi^2 u}f(u)\,du.$$

Since $v(x,0) = 0$, $V_s(\xi,0) = 0$, and hence $A(\xi) = 0$. Hence

$$v(x,t) = \frac{2}{\pi}\int\limits_0^\infty \xi e^{-\xi^2 t}\sin\xi x\,d\xi \int\limits_0^t e^{\xi^2 u}f(u)\,du$$

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† Riemann-Weber, 2, § 40; Carslaw, *Heat*, § 23.

$$= \frac{2}{\pi}\int\limits_0^t f(u)\,du\int\limits_0^\infty \xi e^{\xi^2(u-t)}\sin\xi x\,d\xi$$

$$= \frac{x}{2\surd\pi}\int\limits_0^t f(u)(t-u)^{-\frac{3}{2}}e^{-\frac{1}{4}x^2/(t-u)}\,du.$$

That this tends to $f(x)$ in general follows from Theorem 13.

For a rigorous proof suppose again that $v(x,t)$ is of exponential type. Let

$$V(\zeta,t) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty v(x,t)e^{i\zeta x}\,dx \quad (\eta > c).$$

Then

$$\sqrt{(2\pi)}\frac{\partial V}{\partial t} = \int\limits_0^\infty \frac{\partial v}{\partial t}e^{i\zeta x}\,dx = \int\limits_0^\infty \frac{\partial^2 v}{\partial x^2}e^{i\zeta x}\,dx$$

$$= -v_x(0,t)+i\zeta f(t)-\sqrt{(2\pi)}\zeta^2 V,$$

integrating by parts twice. Hence

$$V(\zeta,t) = A(\zeta)e^{-\zeta^2 t}+\frac{e^{-\zeta^2 t}}{\sqrt{(2\pi)}}\int\limits_0^t \{i\zeta f(u)-v_x(0,u)\}e^{\zeta^2 u}\,du.$$

As before, $A(\zeta) = 0$; if $f(u)$ and $v_x(0,u)$ are bounded in a finite interval, the other term is

$$O\left(e^{-\xi^2 t}\int\limits_0^t |\xi|e^{\xi^2 u}\,du\right) = O(|\xi|^{-1})$$

as $\xi \to \pm\infty$. Hence, by L^2 theory,

$$v(x,t) = \frac{1}{2\pi}\frac{d}{dx}\int\limits_{ia-\infty}^{ia+\infty}\frac{e^{-i\zeta x}-1}{-i\zeta}e^{-\zeta^2 t}\,d\zeta\int\limits_0^t \{i\zeta f(u)-v_x(0,u)\}e^{\zeta^2 u}\,du$$

for $x > 0$, while the right-hand side is 0 for $x < 0$. The repeated integral is absolutely convergent, and we may invert, and then replace a by 0. The term in $f(u)$ contributes

$$\frac{1}{2\pi}\frac{d}{dx}\int\limits_0^t f(u)\,du\int\limits_{-\infty}^\infty (1-e^{-i\xi x})e^{\xi^2(u-t)}\,d\xi$$

$$= \frac{1}{2\surd\pi}\frac{d}{dx}\int\limits_0^t f(u)\{1-e^{-\frac{1}{4}x^2/(t-u)}\}\frac{du}{\sqrt{(t-u)}}$$

$$= \frac{x}{4\surd\pi}\int\limits_0^t f(u)e^{-\frac{1}{4}x^2/(t-u)}\frac{du}{(t-u)^{\frac{3}{2}}}.$$

The contribution of the other term is an even function of x. Changing the sign of x and subtracting, we obtain the same result as before.

10.8. *Obtain the solution of*

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-\alpha^2 v,$$

where $v(x,0)=0$ $(x>0)$, $v(0,t)=f(t)$ $(t>0)$.

Proceeding as before, we obtain

$$\frac{\partial V_s}{\partial t}=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty\left(\frac{\partial^2 v}{\partial x^2}-\alpha^2 v\right)\sin\xi x\,dx$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}\xi f(t)-\xi^2 V_s-\alpha^2 V_s.$$

Hence

$$V_s=A(\xi)e^{-(\xi^2+\alpha^2)t}+\sqrt{\left(\frac{2}{\pi}\right)}\xi e^{-(\xi^2+\alpha^2)t}\int\limits_0^t e^{(\xi^2+\alpha^2)u}f(u)\,du,$$

$A(\xi)=0$ as before, and

$$v(x,t)=\frac{2}{\pi}\int\limits_0^t f(u)\,du\int\limits_0^\infty \xi e^{(\xi^2+\alpha^2)(u-t)}\sin\xi x\,d\xi$$

$$=\frac{x}{2\sqrt{\pi}}\int\limits_0^t f(u)(t-u)^{-\frac{3}{2}}e^{-\alpha^2(t-u)-\frac{1}{4}x^2/(t-u)}\,du.$$

The rigorous solution may be obtained as before.

10.9. *Solve*†

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-\alpha^2 v\quad(0<x<l),$$

where $v=v_0$ $(x=0,\ t>0)$, $v=0$ $(t=0,\ 0<x<l)$, *and*

$$\frac{\partial v}{\partial x}=0\quad(x=l).$$

Here we take t as the variable of the Fourier integral, and suppose that $|v(x,t)|<Ke^{c|t|}$ for all x. Let

$$V(x,\zeta)=\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty v(x,t)e^{i\zeta t}\,dt\quad(\eta>c).$$

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† See Jeffreys, *Operational Methods*, p. 70.

Then

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \frac{\partial^2 v}{\partial x^2} e^{i\zeta t}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \left(\frac{\partial v}{\partial t}+\alpha^2 v\right)e^{i\zeta t}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}}\left[ve^{i\zeta t}\right]_0^\infty - \frac{i\zeta}{\sqrt{(2\pi)}}\int\limits_0^\infty ve^{i\zeta t}\,dt + \frac{\alpha^2}{\sqrt{(2\pi)}}\int\limits_0^\infty ve^{i\zeta t}\,dt$$

$$= (\alpha^2 - i\zeta)V.$$

Hence $V = A(\zeta)\cosh\{\sqrt{(\alpha^2-i\zeta)}x\}+B(\zeta)\sinh\{\sqrt{(\alpha^2-i\zeta)}x\}$.

When $x = 0$,

$$V(0,\zeta) = \frac{v_0}{\sqrt{(2\pi)}}\int\limits_0^\infty e^{i\zeta t}\,dt = -\frac{v_0}{i\zeta\sqrt{(2\pi)}}.$$

Hence

$$A(\zeta) = -\frac{v_0}{i\zeta\sqrt{(2\pi)}}.$$

When $x = l$, $\dfrac{\partial V}{\partial x} = 0$. Hence

$$A(\zeta)\sinh\{\sqrt{(\alpha^2-i\zeta)}l\}+B(\zeta)\cosh\{\sqrt{(\alpha^2-i\zeta)}l\} = 0.$$

Hence

$$V = -\frac{v_0}{i\zeta\sqrt{(2\pi)}}\left[\cosh\{\sqrt{(\alpha^2-i\zeta)}x\} - \frac{\sinh\{\sqrt{(\alpha^2-i\zeta)}l\}\sinh\{\sqrt{(\alpha^2-i\zeta)}x\}}{\cosh\{\sqrt{(\alpha^2-i\zeta)}l\}}\right]$$

$$= -\frac{v_0}{i\zeta\sqrt{(2\pi)}}\frac{\cosh\{\sqrt{(\alpha^2-i\zeta)}(x-l)\}}{\{\cosh\sqrt{(\alpha^2-i\zeta)}l\}}.$$

Hence for $t > 0$

$$v(x,t) = -\frac{v_0}{2\pi i}\int\limits_{ia-\infty}^{ia+\infty} \frac{\cosh\{\sqrt{(\alpha^2-i\zeta)}(x-l)\}}{\cosh\{\sqrt{(\alpha^2-i\zeta)}l\}}e^{-i\zeta t}\frac{d\zeta}{\zeta}.$$

Here $\arg\sqrt{(\alpha^2-i\zeta)}$ varies from $\frac{1}{4}\pi$ to $-\frac{1}{4}\pi$, and the integral is absolutely convergent if $0 < x < l$.

10.10. *Obtain the solution of*[†]

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r} \quad (t > 0,\ r > a)$$

such that $v(r,0) = 0$ $(r > a)$, $v(a,t) = f(t)$.

[†] See Nicholson (2), Goldstein (2).

Suppose that $v(r,t) = O(e^{cr+ct})$. If

$$V(r,\zeta) = \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty v(r,t)e^{i\zeta t}\,dt \quad (\eta > c),$$

then

$$\sqrt{(2\pi)}\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right) = \int\limits_0^\infty \frac{\partial v}{\partial t}e^{i\zeta t}\,dt$$

$$= \big[ve^{i\zeta t}\big]_0^\infty - i\zeta\int\limits_0^\infty ve^{i\zeta t}\,dt$$

$$= -i\zeta\sqrt{(2\pi)}V.$$

The solution of this may be written

$$V(r,\zeta) = A(\zeta)H_0^{(1)}\{r\sqrt{(i\zeta)}\}+B(\zeta)H_0^{(2)}\{r\sqrt{(i\zeta)}\},$$

where

$$H_0^{(1)}(z) = J_0(z)+iY_0(z), \qquad H_0^{(2)}(z) = J_0(z)-iY_0(z).$$

Let $\zeta = \xi+ik$, $\sqrt{(i\zeta)} = \zeta' = \xi'+i\eta'$. Then

$$\xi'^2-\eta'^2 = -k,$$

i.e. ζ' varies along a branch of this rectangular hyperbola, say the upper branch. On it

$$|H_0^{(1)}(r\zeta')| \sim \frac{Ae^{-r\eta'}}{\sqrt{(|r\zeta'|)}}, \qquad |H_0^{(2)}(r\zeta')| \sim \frac{Ae^{r\eta'}}{\sqrt{(|r\zeta'|)}}.$$

Since $V(r,\zeta) = O(e^{cr})$ in the upper half-plane, we must have $B(\zeta) = 0$.

Also, as $r \to a$,

$$V(r,\zeta) \to \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty f(t)e^{i\zeta t}\,dt = F(\zeta).$$

Hence

$$A(\zeta) = \frac{F(\zeta)}{H_0^{(1)}\{a\sqrt{(i\zeta)}\}},$$

and the solution is

$$v(r,t) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ik-\infty}^{ik+\infty} F(\zeta)\frac{H_0^{(1)}\{r\sqrt{(i\zeta)}\}}{H_0^{(1)}\{a\sqrt{(i\zeta)}\}}e^{-i\zeta t}\,d\zeta,$$

where $\frac{1}{4}\pi \leqslant \arg\sqrt{(i\zeta)} \leqslant \frac{3}{4}\pi$.

For suitable functions $f(t)$ we can make $k \to 0$, and obtain the solution in the form of integrals along the real axis.

10.11. *Obtain the solution of*

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad (x>0,\ 0<y<b)$$

such that

$$\begin{aligned} v &= f(x) && (y=0,\ 0<x<\infty),\\ v &= 0 && (y=b,\ 0<x<\infty),\\ v &= 0 && (x=0,\ 0<y<b). \end{aligned}$$

This is the problem of the steady distribution of heat in a semi-infinite strip with the edges kept at given temperatures.†

Formally, let

$$V_s(\xi,y)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty v(x,y)\sin\xi x\,dx.$$

Then

$$\begin{aligned}\frac{\partial^2 V_s}{\partial y^2} &= \sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty \frac{\partial^2 v}{\partial y^2}\sin\xi x\,dx\\ &= -\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty \frac{\partial^2 v}{\partial x^2}\sin\xi x\,dx\\ &= -\sqrt{\left(\frac{2}{\pi}\right)}\left[\frac{\partial v}{\partial x}\sin\xi x\right]_0^\infty+\sqrt{\left(\frac{2}{\pi}\right)}\xi\int\limits_0^\infty \frac{\partial v}{\partial x}\cos\xi x\,dx\\ &= \left[\sqrt{\left(\frac{2}{\pi}\right)}\xi v\cos\xi x\right]_0^\infty+\sqrt{\left(\frac{2}{\pi}\right)}\xi^2\int\limits_0^\infty v\sin\xi x\,dx\\ &= \xi^2 V_s.\end{aligned}$$

Hence $$V_s=A(\xi)\cosh\xi y+B(\xi)\sinh\xi y.$$

Making $y\to 0$, $$A(\xi)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty f(x)\sin\xi x\,dx,$$

so that $A(\xi)$ is the sine transform of $f(x)$, $A(\xi)=F_s(\xi)$.

Putting $y=b$,

$$A(\xi)\cosh\xi b+B(\xi)\sinh\xi b=0,$$

$$B(\xi)=-\coth\xi b\,F_s(\xi).$$

Hence

$$\begin{aligned}V_s &= F_s(\xi)(\cosh\xi y-\sinh\xi y\coth\xi b)\\ &= F_s(\xi)\frac{\sinh\xi(b-y)}{\sinh\xi b},\end{aligned}$$

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† Carslaw, *Heat*, § 45.

and $$v(x,y) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty F_s(\xi)\frac{\sinh \xi(b-y)}{\sinh \xi b}\sin \xi x\, d\xi.$$

In terms of f this gives

$$v(x,y) = \frac{2}{\pi}\int_0^\infty f(u)\,du \int_0^\infty \frac{\sinh\xi(b-y)}{\sinh\xi b}\sin\xi x \sin\xi u\, d\xi$$

$$= \frac{1}{2b}\sin\frac{\pi y}{b}\int_0^\infty f(u)\left(\frac{1}{\cos(b-y)\pi/b+\cosh(x-u)\pi/b}- \right.$$

$$\left. -\frac{1}{\cos(b-y)\pi/b+\cosh(x+u)\pi/b}\right) du.$$

That this tends to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ wherever it exists, follows from Theorem 18; for

$$\frac{\sinh\xi(b-y)}{\sinh\xi b} = e^{-\xi y}-e^{-\xi b}\frac{\sinh\xi y}{\sinh\xi b},$$

and the contribution of the last term is clearly 0.

Suppose now that $v(x,y) = O(e^{cx})$ as $x \to \infty$, uniformly with respect to y, where $\pi/b < c < 2\pi/b$.

Let $$V(\zeta,y) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty v(x,y)e^{i\zeta x}\,dx,$$

where $c < \eta < 2\pi/b$. Then

$$\sqrt{(2\pi)}\frac{\partial^2 V}{\partial y^2} = \int_0^\infty \frac{\partial^2 v}{\partial y^2}e^{i\zeta x}\,dx$$

$$= -\int_0^\infty \frac{\partial^2 v}{\partial x^2}e^{i\zeta x}\,dx$$

$$= -\left[\frac{\partial v}{\partial x}e^{i\zeta x}\right]_0^\infty + i\zeta[ve^{i\zeta x}]_0^\infty + \zeta^2\int_0^\infty ve^{i\zeta x}\,dx$$

$$= g(y)+\sqrt{(2\pi)}\zeta^2 V,$$

where $g(y) = v_x(0,y)$. Hence

$$V(\zeta,y) = A(\zeta)\cosh\zeta y + B(\zeta)\sinh\zeta y + \frac{1}{\sqrt{(2\pi)}\zeta}\int_0^y \sinh\zeta(y-u)\,g(u)\,du.$$

Making $y \to 0$, we obtain, by dominated convergence,

$$A(\zeta) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty f(x)e^{i\zeta x}\,dx = F(\zeta),$$

and making $y \to b$,

$$0 = A(\zeta)\cosh\zeta b + B(\zeta)\sinh\zeta b + \frac{1}{\sqrt{(2\pi)\zeta}}\int\limits_0^b \sinh\zeta(b-u)\,g(u)\,du.$$

Hence

$$V(\zeta,y) = F(\zeta)\frac{\sinh\zeta(b-y)}{\sinh\zeta b} + \frac{\sinh\zeta(y-b)}{\sqrt{(2\pi)\zeta}\sinh\zeta b}\int\limits_0^y \sinh\zeta u\, g(u)\,du -$$

$$-\frac{\sinh\zeta y}{\sqrt{(2\pi)\zeta}\sinh\zeta b}\int\limits_y^b \sinh\zeta(b-u)\,g(u)\,du.$$

Hence if $c < a < 2\pi/b$,

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F(\zeta)\frac{\sinh\zeta(b-y)}{\sinh\zeta b}e^{-i\zeta x}\,d\zeta +$$

$$+\frac{1}{2\pi}\int\limits_0^y g(u)\,du \int\limits_{ia-\infty}^{ia+\infty} \frac{\sinh\zeta(y-b)\sinh\zeta u}{\zeta\sinh\zeta b}e^{-i\zeta x}\,d\zeta -$$

$$-\frac{1}{2\pi}\int\limits_y^b g(u)\,du \int\limits_{ia-\infty}^{ia+\infty} \frac{\sinh\zeta y\sinh\zeta(b-u)}{\zeta\sinh\zeta b}e^{-i\zeta x}\,d\zeta$$

$$= v(x,y) \quad (x > 0), \qquad 0 \quad (x < 0).$$

Replacing x by $-x$ and subtracting, we find that, for $x > 0$, $v(x,y)$ is equal to the above expression with $e^{-i\zeta x}$ replaced by $-2i\sin\zeta x$. In this form, if we replace a by 0 in the last two terms, we obtain 0; but if $a > \pi/b$, the pole at $\zeta = i\pi/b$ gives a residue term of the form

$$K\sin\frac{\pi y}{b}\sinh\frac{\pi x}{b}. \tag{10.11.1}$$

In the first term we may insert the Fourier integral for $F(\zeta)$ and invert, by absolute convergence. The result (again allowing for the pole at $\zeta = i\pi/b$) is

$$\int\limits_0^\infty f(u)\chi(u)\,du,$$

where

$$\chi(u) = \frac{1}{2b}\sin\frac{\pi y}{b}\left(\frac{1}{\cos\frac{b-y}{b}\pi+\cosh\frac{x-u}{b}\pi} - \frac{1}{\cos\frac{b-y}{b}\pi+\cosh\frac{x+u}{b}\pi}\right)$$

$$-\frac{2}{b}\sin\frac{\pi y}{b}\sinh\frac{\pi x}{b}e^{-\pi u/b}.$$

We obtain the same solution as before, plus a term of the form (10.11.1), which is a solution of the corresponding problem with $f = 0$.

The solution in this case is therefore not unique, unless we make some hypothesis which excludes such a term.

10.12. *Obtain the solution of*

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial x^2}\quad(-\infty<x<\infty,\ t>0),$$

such that $y(x,0)=f(x)$, $y_t(x,0)=g(x)$.

This is the problem of the motion of an infinite string with a given initial displacement and velocity, y being the displacement at distance x along the string at time t.

For a formal solution, let

$$Y(\xi,t)=\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} y(x,t)e^{i\xi x}\,dx.$$

Then

$$\begin{aligned}\frac{\partial^2 Y}{\partial t^2}&=\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{i\xi x}\,dx\\&=\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{i\xi x}\,dx\\&=-\xi^2 Y,\end{aligned}$$

integrating by parts twice. Hence

$$Y=A(\xi)\cos\xi t+B(\xi)\sin\xi t,$$

and clearly $A(\xi)=F(\xi)$, $\xi B(\xi)=G(\xi)$. Hence

$$y(x,t)=\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}F(\xi)\cos\xi t\,e^{-i\xi x}\,d\xi+\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{G(\xi)}{\xi}\sin\xi t\,e^{-i\xi x}\,d\xi.$$

The first term is

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\{e^{-i\xi(x-t)}+e^{-i\xi(x+t)}\}\,d\xi\int\limits_{-\infty}^{\infty}f(u)e^{i\xi u}\,du=\tfrac{1}{2}\{f(x-t)+f(x+t)\},$$

and the second is

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\sin\xi t}{\xi}e^{-i\xi x}\,d\xi\int\limits_{-\infty}^{\infty}g(u)e^{i\xi u}\,du\\&\qquad=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}g(u)\,du\int\limits_{0}^{\infty}\frac{\sin\xi t\cos\xi(x-u)}{\xi}\,d\xi=\tfrac{1}{2}\int\limits_{x-t}^{x+t}g(u)\,du.\end{aligned}$$

Hence $$y(x,t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}+\tfrac{1}{2}\int\limits_{x-t}^{x+t}g(u)\,du,$$

the classical solution.

For a rigorous solution let $y(x,t)=O(e^{c|x|})$, and similarly for the partial derivatives. Let

$$Y_+(\zeta,t)=\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty y(x,t)e^{i\zeta x}\,dx,\qquad Y_-(\zeta,t)=\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^0 y(x,t)e^{i\zeta x}\,dx,$$

where Y_+ exists for $\eta>c$, Y_- for $\eta<-c$. Now

$$\sqrt{(2\pi)}\frac{\partial^2Y_+}{\partial t^2}=\int\limits_0^\infty\frac{\partial^2y}{\partial t^2}e^{i\zeta x}\,dx=\int\limits_0^\infty\frac{\partial^2y}{\partial x^2}e^{i\zeta x}\,dx$$

$$=\left[\frac{\partial y}{\partial x}e^{i\zeta x}\right]_0^\infty-i\zeta[ye^{i\zeta x}]_0^\infty-\zeta^2\int\limits_0^\infty ye^{i\zeta x}\,dx$$

$$=-\phi(t)+i\zeta\psi(t)-\zeta^2\sqrt{(2\pi)}Y_+,$$

where $\phi(t)=y_x(0,t)$, $\psi(t)=y(0,t)$. Hence

$$Y_+=A(\zeta)\cos\zeta t+B(\zeta)\sin\zeta t-\frac{1}{\sqrt{(2\pi)}\zeta}\int\limits_0^t\sin\zeta(t-u)\{\phi(u)-i\zeta\psi(u)\}\,du,$$

and the initial conditions give $A(\zeta)=F_+(\zeta)$, $\zeta B(\zeta)=G_+(\zeta)$. Then

$$Y_+(\zeta,t)=F_+(\zeta)\cos\zeta t+\zeta^{-1}G_+(\zeta)\sin\zeta t+\chi(\zeta,t),$$

where χ is an integral function which tends to 0 for $\zeta=\xi+ik$, $\xi\to\pm\infty$. Similarly,

$$Y_-(\zeta,t)=F_-(\zeta)\cos\zeta t+\zeta^{-1}G_-(\zeta)\sin\zeta t-\chi(\zeta,t).$$

Now
$$y(x,t)=\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda}Y_+(\zeta,t)e^{-i\zeta x}\,d\zeta+\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ib-\lambda}^{ib+\lambda}Y_-(\zeta,t)e^{-i\zeta x}\,d\zeta,$$

where $a>c$, $b<-c$. The contribution of $\chi(\zeta,t)$ to this is 0. The contribution of F is

$$\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ia-\lambda}^{ia+\lambda}F_+(\zeta)\tfrac{1}{2}\{e^{-i\zeta(x+t)}+e^{-i\zeta(x-t)}\}\,d\zeta+$$

$$+\frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{ib-\lambda}^{ib+\lambda}F_-(\zeta)\tfrac{1}{2}\{e^{-i\zeta(x+t)}+e^{-i\zeta(x-t)}\}\,d\zeta=\tfrac{1}{2}\{f(x+t)+f(x-t)\}.$$

The contribution of G_+ is

$$\frac{1}{2\pi}\int_0^\infty g(u)\,du\int_{ia-\infty}^{ia+\infty}\frac{\sin\zeta t}{\zeta}e^{i\zeta(u-x)}\,d\zeta$$

(inverting by the bounded convergence of the ζ-integral). The ζ-integral is π if $x-t<u<x+t$, and otherwise 0. Similarly for G_-, and we obtain the same result as before.

10.13 *Obtain the solution of*†

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}=0 \quad (0<x<l,\ t>0)$$

such that $y(0,t)=0$, $y(l,t)=0$, $y(x,0)=f(x)$, *and* $y_t(x,0)=0$.

This is the problem of the vibration of an elastic string with fixed ends, y being the displacement of the string at distance x along the string at time t.

Suppose that $y(x,t)$ *and its derivatives are* $O(e^{ct})$ *for some* c. Let

$$Y(x,\zeta)=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty y(x,t)e^{i\zeta t}\,dt$$

for $\eta>c$. Then

$$\sqrt{(2\pi)}\frac{\partial^2 Y}{\partial x^2}=\int_0^\infty\frac{\partial^2 y}{\partial x^2}e^{i\zeta t}\,dt=\int_0^\infty\frac{\partial^2 y}{\partial t^2}e^{i\zeta t}\,dt$$

$$=\left[\frac{\partial y}{\partial t}e^{i\zeta t}\right]_0^\infty-i\zeta\int_0^\infty\frac{\partial y}{\partial t}e^{i\zeta t}\,dt$$

$$=-i\zeta[ye^{i\zeta t}]_0^\infty-\zeta^2\int_0^\infty ye^{i\zeta t}\,dt=i\zeta f(x)-\zeta^2\sqrt{(2\pi)}Y.$$

Hence

$$Y=A(\zeta)\cos\zeta x+B(\zeta)\sin\zeta x+\frac{i}{\sqrt{(2\pi)}}\int_0^x f(u)\sin\zeta(x-u)\,du.$$

The initial conditions give $Y(0,\zeta)=0$, $Y(l,\zeta)=0$. Hence $A(\zeta)=0$, and

$$B(\zeta)\sin\zeta l+\frac{i}{\sqrt{(2\pi)}}\int_0^l\sin\zeta(l-u)f(u)\,du=0.$$

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† Riemann-Weber, 2, § 85.

Hence

$$Y = -\frac{i}{\sqrt{(2\pi)}}\frac{\sin\zeta x}{\sin\zeta l}\int\limits_0^l f(u)\sin\zeta(l-u)\,du+\frac{i}{\sqrt{(2\pi)}}\int\limits_0^x f(u)\sin\zeta(x-u)\,du$$

$$= -\frac{i}{\sqrt{(2\pi)}}\frac{\sin\zeta(l-x)}{\sin\zeta l}\int\limits_0^x f(u)\sin\zeta u\,du-$$

$$-\frac{i}{\sqrt{(2\pi)}}\frac{\sin\zeta x}{\sin\zeta l}\int\limits_x^l f(u)\sin\zeta(l-u)\,du.$$

Hence

$$y(x,t) = \frac{1}{2\pi i}\int\limits_{ia-\infty}^{ia+\infty}\frac{\sin\zeta(l-x)}{\sin\zeta l}e^{-i\zeta t}\,d\zeta\int\limits_0^x f(u)\sin\zeta u\,du+$$

$$+\frac{1}{2\pi i}\int\limits_{ia-\infty}^{ia+\infty}\frac{\sin\zeta x}{\sin\zeta l}e^{-i\zeta t}\,d\zeta\int\limits_x^l f(u)\sin\zeta(l-u)\,du. \quad (10.13.1)$$

If we replace t by $t+2l$ and subtract, we introduce a factor $2i\sin\zeta l\,e^{-i\zeta l}$, and the resulting integrals tend to 0 when $a \to -\infty$, if $t > 0$. Hence the solution has the period $2l$, and we may suppose $0 < t < 2l$. We then write

$$\frac{1}{\sin\zeta l} = -2ie^{i\zeta l}+\frac{e^{2i\zeta l}}{\sin\zeta l},$$

and the contribution of the last term is seen to be 0 on making $a \to \infty$. The contribution of the first term may then be deduced from Fourier's theorem. For example, the first term in (10.13.1) gives

$$\frac{1}{4\pi}\int\limits_{ia-\infty}^{ia+\infty}\{e^{i\zeta(2l-x-t)}-e^{i\zeta(x-t)}\}\,d\zeta\int\limits_0^x f(u)(e^{i\zeta u}-e^{-i\zeta u})\,du.$$

The first terms in each bracket give

$$-\tfrac{1}{2}f(2l-x-t)$$

if $2l-2x < t < 2l-x$, and otherwise 0. The complete solution may be written

$$\tfrac{1}{2}\{f(x+t)+f(x-t)\},$$

where $f(x)$ is defined outside $(0,l)$ by saying that it is odd and has the period $2l$.

If $f''(x)$ exists everywhere and is continuous, the whole process is plainly valid. In other cases the differential equation is not satisfied everywhere, and the given conditions are not strictly consistent.

Suppose, for example, that $f(x) = x$ $(0 \leqslant x \leqslant \frac{1}{2}l)$, $f(x) = l-x$ $(\frac{1}{2}l \leqslant x \leqslant l)$. Then $f'(x) = 1$ or -1, and $f''(x) = 0$ where it exists.

To cover a case of this kind we could restate the problem by assuming that, instead of the differential equation, y satisfies

$$\left(\frac{\partial y}{\partial x}\right)_{x=x_2} - \left(\frac{\partial y}{\partial x}\right)_{x=x_1} = \frac{\partial^2}{\partial t^2} \int\limits_{x_1}^{x_2} y\,dx$$

for every t, for all but a finite number of values of x, and that $\partial y/\partial x$ is bounded in x for each t. Then

$$\left(\frac{\partial Y}{\partial x}\right)_{x=x_2} - \left(\frac{\partial Y}{\partial x}\right)_{x=x_1} = \frac{1}{\sqrt{(2\pi)}} \int\limits_0^\infty \left\{\left(\frac{\partial y}{\partial x}\right)_{x=x_2} - \left(\frac{\partial y}{\partial x}\right)_{x=x_1}\right\} e^{i\zeta t}\,dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int\limits_0^\infty \left(\frac{\partial^2}{\partial t^2} \int\limits_{x_1}^{x_2} y\,dx\right) e^{i\zeta t}\,dt$$

$$= -\frac{i\zeta}{\sqrt{(2\pi)}} \left[\int\limits_{x_1}^{x_2} y\,dx\, e^{i\zeta t}\right]_0^\infty - \frac{\zeta^2}{\sqrt{(2\pi)}} \int\limits_0^\infty e^{i\zeta t}\,dt \int\limits_{x_1}^{x_2} y\,dx$$

$\left(\text{assuming that } \frac{\partial}{\partial t} \int\limits_{x_1}^{x_2} y\,dx \to 0 \text{ as } t \to 0\right)$

$$= \frac{i\zeta}{\sqrt{(2\pi)}} \int\limits_{x_1}^{x_2} f(x)\,dx - \zeta^2 \int\limits_{x_1}^{x_2} Y\,dx.$$

Hence $\dfrac{\partial^2 Y}{\partial x^2}$ exists, and is equal to $\dfrac{i\zeta f(x)}{\sqrt{(2\pi)}} - \zeta^2 Y$. The analysis then proceeds as before.

Another equation[†] which may be solved in a similar way is

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = g \quad (0 < x < l),$$

where g is a constant, $y(x,0) = g(lx-\frac{1}{2}x^2)$, $y(0,t) = 0$, $y_t(x,0) = 0$.

10.14. The problem[‡] of the waves on a plane sheet of water, caused by a disturbance of strength $f(t)$ at a fixed point (the origin), depends on the solution of

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) \quad (r > 0,\ t > 0),$$

[†] Jeffreys, *Operational Methods*, p. 59.

[‡] Lamb, *Hydrodynamics*, p. 297.

where $$\lim_{r\to 0}\left(-2\pi r\frac{\partial\phi}{\partial r}\right) = f(t) \quad (t > 0).$$

Let $$\Phi(r,\zeta) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \phi(r,t)e^{i\zeta t}\,dt.$$

Then $$c^2\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \frac{\partial^2\phi}{\partial t^2}e^{i\zeta t}\,dt.$$

If the surface of the water is initially flat and at rest, $\phi(r,0) = 0$ and $\phi_t(r,0) = 0$, and the usual partial integration shows that the right-hand side is $-\zeta^2\Phi$. Hence†

$$\Phi(r,\zeta) = A(\zeta)H_0^{(1)}\left(\frac{r\zeta}{c}\right)+B(\zeta)H_0^{(2)}\left(\frac{r\zeta}{c}\right).$$

Since Φ must be bounded for $\mathbf{I}(\zeta) > 0$, $B(\zeta) = 0$. Also

$$F(\zeta) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty f(t)e^{i\zeta t}\,dt = \lim_{r\to 0}\left\{-\sqrt{(2\pi)}r\int_0^\infty \frac{\partial\phi}{\partial r}e^{i\zeta t}\,dt\right\}$$

$$= \lim_{r\to 0}\left(-2\pi r\frac{\partial\Phi}{\partial r}\right) = \lim_{r\to 0}\frac{2\pi r\zeta}{c}A(\zeta)H_1^{(1)}\left(\frac{r\zeta}{c}\right) = 2\pi A(\zeta).$$

Hence $$\Phi(r,\zeta) = \frac{1}{2\pi}F(\zeta)H_0^{(1)}\left(\frac{r\zeta}{c}\right),$$

$$\phi(r,t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{ia-\infty}^{ia+\infty} F(\zeta)H_0^{(1)}\left(\frac{r\zeta}{c}\right)e^{-i\zeta t}\,d\zeta$$

$$= \frac{1}{4\pi^2}\int_0^\infty f(u)\,du\int_{ia-\infty}^{ia+\infty} H_0^{(1)}\left(\frac{r\zeta}{c}\right)e^{i\zeta(u-t)}\,d\zeta.$$

The inner integral is 0 if $t < u+r/c$, and otherwise it is

$$\frac{2\pi}{\sqrt{\{(u-t)^2-r^2/c^2\}}}.$$

Hence
$$\phi(r,t) = \frac{1}{2\pi}\int_0^{t-r/c}\frac{f(u)\,du}{\sqrt{\{(t-u)^2-r^2/c^2\}}} = \frac{1}{2\pi}\int_0^{\cosh^{-1}ct/r} f\left(t-\frac{r}{c}\cosh\lambda\right)d\lambda.$$

10.15. *Obtain the solution of*‡

$$\frac{\partial^2 u}{\partial x\partial y} = u \quad (x > 0,\ y > 0)$$

such that $u(x,0) = a$, $u(0,y) = a$.

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† See Watson, § 3.6.

‡ Bateman, *Partial Differential Equations*, p. 125.

Let
$$U(\zeta,y)=\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty u(x,y)e^{i\zeta x}\,dx$$
for $\eta > c$. Then
$$\frac{\partial U}{\partial y}=\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty \frac{\partial u}{\partial y}e^{i\zeta x}\,dx$$
$$=\frac{1}{\sqrt{(2\pi)}}\left[\frac{\partial u}{\partial y}\frac{e^{i\zeta x}}{i\zeta}\right]_0^\infty-\frac{1}{i\zeta\sqrt{(2\pi)}}\int\limits_0^\infty\frac{\partial^2 u}{\partial x\partial y}e^{i\zeta x}\,dx$$
$$=-\frac{U}{i\zeta},$$
since (with sufficient continuity) $u_y(0,y)=0$. Hence
$$U(\zeta,y)=A(\zeta)e^{iy/\zeta}.$$

Making $y\to 0$,
$$A(\zeta)=\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\infty ae^{i\zeta x}\,dx=-\frac{a}{i\zeta\sqrt{(2\pi)}}.$$

Hence
$$u(x,y)=-\frac{a}{2\pi i}\int\limits_{ik-\infty}^{ik+\infty}\frac{e^{iy/\zeta-ix\zeta}}{\zeta}\,d\zeta$$
$$=aI_0\{2\sqrt{(xy)}\}$$
by (7.13.9).

10.16. Differential-difference equations.† We shall illustrate the general method of solution by considering the simple special case
$$f'(x)=\frac{1}{2h}\{f(x+h)-f(x-h)\}. \qquad (10.16.1)$$

We shall first assume that $f(x)=O(e^{c|x|})$ for some positive c. It follows by repeated appeal to the equation that $f(x)$ has derivatives of all orders, each of which is $O(e^{c|x|})$; and if $f(x)$ satisfies the equation, so does $f^*(x)=f(x)-f(0)-xf'(0)$, and $f^*(0)=f^{*\prime}(0)=0$. Hence we may suppose without loss of generality that $f(0)=f'(0)=0$.

Define $F_+(w)$, $F_-(w)$ as usual, for $v>c$, $v<-c$ respectively. Then
$$F_+(w)=-\frac{1}{\sqrt{(2\pi)}w^2}\int\limits_0^\infty f''(x)e^{ixw}\,dx=O\left(\frac{1}{|w|^3}\right)$$

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† Hilb (2), Titchmarsh (16), Kitagawa (1), Dickson (1).

as $|w| \to \infty$; and similarly for $F_-(w)$. Hence (1.3.4) and the formula obtained from it by differentiating under the integral sign are both valid; and (10.16.1) gives

$$\int\limits_{ia-\infty}^{ia+\infty} \left(-iw - \frac{e^{-ihw} - e^{ihw}}{2h}\right) F_+(w) e^{-ixw}\, dw + \ldots = 0,$$

where $+\ldots$ indicates the corresponding term with b and F_- instead of a and F_+. Hence, by Theorem 141,

$$\left(\frac{\sin hw}{h} - w\right) F_+(w) = \chi(w), \qquad \left(\frac{\sin hw}{h} - w\right) F_-(w) = -\chi(w),$$

where $\chi(w)$ is regular for $b \leqslant v \leqslant a$, and $\chi(w) \to 0$ as $u \to \pm\infty$. Hence

$$f(x) = \frac{h}{\sqrt{(2\pi)}} \int\limits_{ia-\infty}^{ia+\infty} \frac{\chi(w) e^{-ixw}}{\sin hw - hw}\, dw - \frac{h}{\sqrt{(2\pi)}} \int\limits_{ib-\infty}^{ib+\infty} \frac{\chi(w) e^{-ixw}}{\sin hw - hw}\, dw.$$

This is the sum of the residues at poles in the strip $b < v < a$. There is a triple pole at the origin, giving a quadratic in x. The other zeros of the denominator give exponential terms. *Hence*

$$f(x) = A + Bx + Cx^2 + \sum C_\nu e^{-ixw_\nu}, \tag{10.16.2}$$

where A, B, C, C_ν are constants, and w_ν runs through zeros of $\sin hw - hw$ *other than* 0 *such that* $|\mathbf{I}(w_\nu)| \leqslant c$.

If we do not assume that $f(x) = O(e^{c|x|})$, we can proceed as follows.

Let
$$F_{\alpha,\beta}(w) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{\alpha}^{\beta} f(x) e^{iwx}\, dx.$$

Then

$$\int\limits_{\alpha}^{\beta} f(x+h) e^{iwx}\, dx = \int\limits_{\alpha+h}^{\beta+h} f(x) e^{iw(x-h)}\, dx$$
$$= e^{-iwh}\sqrt{(2\pi)}\{F_{\alpha,\beta}(w) + F_{\beta,\beta+h}(w) - F_{\alpha,\alpha+h}(w)\},$$

and similarly with $-h$. Also

$$\int\limits_{\alpha}^{\beta} f'(x) e^{iwx}\, dx = f(\beta) e^{iw\beta} - f(\alpha) e^{iw\alpha} - iw\sqrt{(2\pi)} F_{\alpha,\beta}(w).$$

On multiplying (10.16.1) by e^{iwx} and integrating over (α, β), we thus obtain

$$\sqrt{(2\pi)}\, ih^{-1}(\sin hw - hw) F_{\alpha,\beta}(w) = \Phi_\alpha(w) - \Phi_\beta(w),$$

where

$$\Phi_\alpha(w) = f(\alpha)e^{iw\alpha} - \frac{\sqrt{(2\pi)}}{2h}\{e^{-iwh}F_{\alpha,\alpha+h}(w) - e^{iwh}F_{\alpha-h,\alpha}(w)\},$$

and similarly for $\Phi_\beta(w)$. Now

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{ia-\infty}^{ia+\infty} F_{\alpha,\beta}(w)e^{-iwx}\,dw \quad (\alpha < x < \beta)$$

for any real a. Choose a so that no zero of $\sin hw - hw$ has the imaginary part a, e.g. take a positive and sufficiently small. Then

$$f(x) = \frac{h}{2\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{\Phi_\alpha(w)e^{-iwx}}{\sin hw - hw}\,dw - \frac{h}{2\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{\Phi_\beta(w)e^{-iwx}}{\sin hw - hw}\,dw.$$

It is easily verified that for a fixed β, and $\mathbf{I}(w) = v > 0$,

$$\Phi_\beta(w) = O(e^{v(h-\beta)}).$$

Also we can choose a sequence of contours, e.g. the squares C_n with vertices at $2n\pi h^{-1}(\pm 1 \pm i)$, on which $|\sin hw - hw| > C|w|$. The usual process of contour integration then gives

$$\frac{h}{2\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{\Phi_\beta(w)e^{-iwx}}{\sin hw - hw}\,dw = \sum \frac{\Phi_\beta(w_\nu)e^{-iw_\nu x}}{\cos hw_\nu - 1} \quad (x < \beta - h),$$

where w_ν runs through the zeros of $\sin hw - hw$ in the upper half-plane.

The coefficients in this series are independent of β, since

$$\frac{\partial \Phi_\beta(w_\nu)}{\partial \beta} = e^{iw_\nu\beta}\left\{f'(\beta) - \frac{1}{2h}f(\beta+h) + \frac{1}{2h}f(\beta-h)\right\} +$$
$$+ e^{iw_\nu\beta}\left(iw_\nu + \frac{1}{2h}e^{-ihw_\nu} - \frac{1}{2h}e^{ihw_\nu}\right) = 0.$$

Similarly the term involving Φ_α gives a series depending on the zeros of $\sin hw - hw$ in the lower half-plane, convergent for $x > \alpha + h$, together with a quadratic in x arising from the triple zero at $w = 0$. The result is that (10.16.2) again holds, w_ν now running through all zeros of $\sin hw - hw$ except $w = 0$, and the series converging uniformly in any finite interval.

10.17. The equation†

$$f^{(n)}(x) + \sum_{\nu=0}^{n-1} a_\nu f^{(\nu)}(x+b_\nu) = g(x)$$

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† For another method see Schmidt (1).

can be treated in a similar way. Consider the case in which each function is $O(e^{c|x|})$. By putting

$$f^*(x) = f(x)-f(0)-...-\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)$$

we can reduce it to a similar problem in which

$$f(0) = f'(0) = ... = f^{(n-1)}(0) = 0;$$

and repeated partial integration then shows that $w^n F_+(w)$ belongs to $L^2(ia-\infty, ia+\infty)$ if $a > c$, and similarly for $F_-(w)$. It follows as in the previous section that

$$\int_{ia-\infty}^{ia+\infty} \{F_+(w)K(w)-G_+(w)\}e^{-ixw}\,dw+... = 0,$$

where
$$K(w) = (-iw)^n+\sum_{\nu=0}^{n-1} a_\nu(-iw)^\nu e^{-ib_\nu w},$$

and the integrals are mean-square integrals. Hence, by Theorem 141,

$$F_+(w)K(w)-G_+(w) = \chi(w), \qquad F_-(w)K(w)-G_-(w) = -\chi(w)$$

where $\chi(w)$ is regular for $b \leqslant v \leqslant a$. Hence

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} \frac{G_+(w)+\chi(w)}{K(w)}e^{-ixw}\,dw+$$
$$+\frac{1}{\sqrt{(2\pi)}}\int_{ib-\infty}^{ib+\infty} \frac{G_-(w)-\chi(w)}{K(w)}e^{-ixw}\,dw,$$

where a and b can be chosen so that all the zeros of $K(w)$ in

$$-c \leqslant v \leqslant c,$$

but no others, lie in $b \leqslant v \leqslant a$. The terms involving $\chi(w)$ can be calculated by the theorem of residues. The result is

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} \frac{G_+(w)}{K(w)}e^{-ixw}\,dw+\frac{1}{\sqrt{(2\pi)}}\int_{ib-\infty}^{ib+\infty} \frac{G_-(w)}{K(w)}e^{-ixw}\,dw+$$
$$+\sum C_\nu e^{-ixw_\nu},$$

where w_ν runs through the zeros of $K(w)$ in the strip $-c \leqslant v \leqslant c$, and C_ν is a constant for simple zeros, a linear function of x for double zeros, and so on.

We have used L^2 theory in the proof, but there is no difficulty in avoiding it, e.g. by first integrating twice, so that all the integrals dealt with are absolutely convergent.

The problem can also be solved by the method of the last section in the case in which the functions are not $O(e^{c|x|})$.

10.18. Difference equations. A pure difference equation can be solved in the same way. Take, for example,

$$f(x+1)-f(x) = g(x)$$

with the usual assumptions about f and g. This is equivalent to

$$\int\limits_{ia-\infty}^{ia+\infty} \{F_+(w)e^{-i(x+1)w}-F_+(w)e^{-ixw}-G_+(w)e^{-ixw}\}\,dw +$$
$$+\int\limits_{ib-\infty}^{ib+\infty} \{F_-(w)e^{-i(x+1)w}-F_-(w)e^{-ixw}-G_-(w)e^{-ixw}\}\,dw = 0.$$

Hence

$$F_+(w)(e^{-iw}-1)-G_+(w) = \chi(w),$$
$$F_-(w)(e^{-iw}-1)-G_-(w) = -\chi(w),$$

where $\chi(w)$ is regular for $b < v < a$. Hence

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} \frac{\chi(w)+G_+(w)}{e^{-iw}-1}e^{-ixw}\,dw -$$
$$-\frac{1}{\sqrt{(2\pi)}}\int\limits_{ib-\infty}^{ib+\infty} \frac{\chi(w)-G_-(w)}{e^{-iw}-1}e^{-ixw}\,dw.$$

The terms involving $\chi(w)$ merely represent a function of period 1, which is obviously part of the solution. Hence the solution is

$$f(x) = f^*(x)+\frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} \frac{G_+(w)}{e^{-iw}-1}e^{-ixw}\,dw +$$
$$+\frac{1}{\sqrt{(2\pi)}}\int\limits_{ib-\infty}^{ib+\infty} \frac{G_-(w)}{e^{-iw}-1}e^{-ixw}\,dw,$$

where $f^*(x)$ is any function of period 1.

The formulae are valid in the L^2 sense if $g(x)e^{-c|x|}$ belongs to L^2 for some c. Under more special circumstances we can reduce it to other forms. If we expand $1/(e^{-iw}-1)$ in powers of e^{-iw}, we obtain formally

$$f(x) = f^*(x)-g(x)-g(x+1)-\dots,$$

which is obviously a solution if the series converges.

XI

INTEGRAL EQUATIONS

11.1. Introduction. THE most familiar form of integral equation is

$$f(x) = g(x)+\lambda \int_{a}^{b} k(x,y)f(y)\,dy,$$

where $g(x)$ and $k(x,y)$ are given functions, and $f(x)$ is to be determined.

The equation can be solved by means of Fourier integrals in certain special cases; these are, roughly, the cases in which $k(x,y)$ is of such a form that the integral is a 'resultant' of one kind or another.

We shall usually suppress the factor λ, which is of no importance in most of our results.

First take $k(x,y) = k(x-y)$, and the limits $-\infty$, ∞, so that the equation is

$$f(x) = g(x)+ \int_{-\infty}^{\infty} k(x-y)f(y)\,dy \quad (-\infty < x < \infty). \tag{11.1.1}$$

A formal solution may be obtained as follows. With our standard notation for transforms, we have

$$\begin{aligned} F(u) &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left\{g(x)+ \int_{-\infty}^{\infty} k(x-y)f(y)\,dy\right\}e^{ixu}\,dx \\ &= G(u)+ \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} k(x-y)e^{ixu}\,dx \\ &= G(u)+ \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} k(t)e^{i(y+t)u}\,dt \\ &= G(u)+\sqrt{(2\pi)}F(u)K(u). \end{aligned} \tag{11.1.2}$$

Hence

$$F(u) = \frac{G(u)}{1-\sqrt{(2\pi)}K(u)}, \tag{11.1.3}$$

and the solution may be written

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{G(u)}{1-\sqrt{(2\pi)}K(u)} e^{-ixu}\,du. \tag{11.1.4}$$

Also, (11.1.4) gives

$$f(x)-g(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\left\{\frac{G(u)}{1-\sqrt{(2\pi)}K(u)}-G(u)\right\}e^{-ixu}\,du$$

$$= \int_{-\infty}^{\infty} G(u)\frac{K(u)}{1-\sqrt{(2\pi)}K(u)}e^{-ixu}\,du.$$

If

$$M(u) = \frac{K(u)}{1-\sqrt{(2\pi)}K(u)},$$

and $m(x)$ is the transform of $M(u)$, this gives

$$f(x) = g(x)+\int_{-\infty}^{\infty} g(t)m(x-t)\,dt \tag{11.1.5}$$

as another formal solution.

The equation

$$f(x) = g(x)+\int_{0}^{\infty} f(y)k\left(\frac{x}{y}\right)\frac{dy}{y} \tag{11.1.6}$$

may be reduced to the form (11.1.1), or solved similarly by Mellin integrals. The formal process is

$$\mathfrak{F}(s) = \mathfrak{G}(s)+\int_{0}^{\infty} x^{s-1}\,dx\int_{0}^{\infty} f(y)k\left(\frac{x}{y}\right)\frac{dy}{y}$$

$$= \mathfrak{G}(s)+\int_{0}^{\infty} f(y)\frac{dy}{y}\int_{0}^{\infty} k\left(\frac{x}{y}\right)x^{s-1}\,dx$$

$$= \mathfrak{G}(s)+\int_{0}^{\infty} f(y)y^{s-1}\,dy\int_{0}^{\infty} k(u)u^{s-1}\,du$$

$$= \mathfrak{G}(s)+\mathfrak{F}(s)\mathfrak{K}(s),$$

and the solution is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathfrak{G}(s)}{1-\mathfrak{K}(s)}x^{-s}\,ds. \tag{11.1.7}$$

This can also be reduced to a form corresponding to (11.1.5).

The simplest conditions under which the process is valid are given by

THEOREM 145. *Let $g(x)$ belong to $L^2(-\infty,\infty)$, and $k(x)$ to $L(-\infty,\infty)$, and let the upper bound of $K(u)$ be less than $1/\sqrt{(2\pi)}$. Then* (11.1.4) *gives a solution of the equation of the class L^2, and any other solution of L^2 is equal to it almost everywhere.*

Clearly $G(u)/\{1-\sqrt{(2\pi)}K(u)\}$ belongs to L^2, so that (11.1.4) exists in the L^2 sense, and defines a function $f(x)$ of L^2; as in § 3.13

$$h(x) = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$$

exists for almost all x, and belongs to L^2; and, by Theorem 65, if F, H, K are the transforms of f, h, k,

$$H(u) = \sqrt{(2\pi)}F(u)K(u) = \frac{\sqrt{(2\pi)}G(u)K(u)}{1-\sqrt{(2\pi)}K(u)}.$$

Hence the transform of $g(x)+h(x)$ is

$$G(u)+\frac{\sqrt{(2\pi)}G(u)K(u)}{1-\sqrt{(2\pi)}K(u)} = \frac{G(u)}{1-\sqrt{(2\pi)}K(u)} = F(u).$$

Hence
$$g(x)+h(x) = f(x)$$
almost everywhere, i.e. the equation is satisfied.

Conversely, if f and g are L^2, k is L, and (11.1.1) holds, then by Theorem 65 (11.1.2) holds, and hence (11.1.4). This proves the theorem.

If also k is L^2, so are K and M, and (11.1.5) is equivalent to (11.1.4).

11.2. The homogeneous equation. We have shown that, so far as the class L^2 goes, the solution is unique. But under special circumstances there may be other solutions not of L^2. If there were two solutions of (11.1.1), their difference would satisfy the homogeneous equation

$$f(x) = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy \quad (-\infty < x < \infty). \tag{11.2.1}$$

This equation is satisfied formally by putting $f(x) = e^{ax}$, if a is such that

$$\int_{-\infty}^{\infty} k(t)e^{-at}\,dt = 1. \tag{11.2.2}$$

We shall next show that, under fairly simple conditions, the only solutions of the homogeneous equation are of this type.

THEOREM 146. *Let* $0 < c < c'$, *and let* $e^{c'|x|}k(x)$ *belong to* L *and* $e^{-c|x|}f(x)$ *to* $L^2(-\infty,\infty)$. *Then, if* $f(x)$ *satisfies* (11.2.1), *it is of the form*

$$f(x) = \sum_{\nu}\sum_{p=1}^{q} C_{\nu,p}\,x^{p-1}e^{-iw_\nu x}, \tag{11.2.3}$$

where w_ν *runs through all the zeros of* $1-\sqrt{(2\pi)}K(w)$ *such that* $|\mathrm{I}(w_\nu)| \leqslant c$, *the* $C_{\nu,p}$ *are constants, and* q *is the order of multiplicity of the zero* w_ν.

It is at once verified that (11.2.3) is a solution of (11.2.1).

To prove the theorem we observe that, with the usual notation, $K(w)$ is analytic for $-c' < v < c'$, $F_+(w)$ is analytic for $v > c$, and $F_-(w)$ analytic for $v < -c$. For $c < a < c'$

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_+(w)e^{-ixw}\,dw = f(x) \quad (x > 0), \qquad 0 \quad (x < 0)$$

in the mean-square sense; and by Theorem 65

$$\int\limits_0^\infty k(x-y)f(y)\,dy = \int\limits_{ia-\infty}^{ia+\infty} F_+(w)K(w)e^{-ixw}\,dw,$$

also in the mean-square sense. Similarly for $F_-(w)$, with a replaced by b, where $-c' < b < -c$. Hence (11.2.1) gives

$$\int\limits_{ia-\infty}^{ia+\infty} F_+(w)\{1-\sqrt{(2\pi)}K(w)\}e^{-ixw}\,dw + {}$$
$$+\int\limits_{ib-\infty}^{ib+\infty} F_-(w)\{1-\sqrt{(2\pi)}K(w)\}e^{-ixw}\,dw = 0$$

in the mean-square sense.

It therefore follows from Theorem 141 that $F_+(w)\{1-\sqrt{(2\pi)}K(w)\}$ and $F_-(w)\{1-\sqrt{(2\pi)}K(w)\}$ can both be continued throughout the strip $b < v < a$, and $F_+(w) = -F_-(w)$ in this strip. Hence $F_+(w)$ and $F_-(w)$ are regular in the strip except possibly for poles at the zeros of $1-\sqrt{(2\pi)}K(w)$.

We can now write

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_+(w)e^{-ixw}\,dw - \frac{1}{\sqrt{(2\pi)}}\int\limits_{ib-\infty}^{ib+\infty} F_+(w)e^{-ixw}\,dw,$$

and, since $F_+(w) \to 0$ as $u \to \pm\infty$, we can evaluate the right-hand side by the calculus of residues in the usual way. This proves the theorem.

In particular, the result is true if $k(x) = O(e^{-c'|x|})$ and $f(x) = O(e^{c|x|})$, where $0 < c < c'$. In this case it can be obtained without recourse to L^2 theory. For, if $c < a < \eta$,

$$\lim_{T\to\infty}\int\limits_{ia-T}^{ia+T} \frac{F_+(w)}{w-\zeta}\,dw = \lim_{T\to\infty}\frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-T}^{ia+T} \frac{dw}{w-\zeta}\int\limits_0^\infty f(x)e^{ixw}\,dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_0^\infty f(x)\,dx \lim_{T\to\infty} \int_{ia-T}^{ia+T} \frac{e^{ixw}}{w-\zeta}\,dw$$

$$= \sqrt{(2\pi)}i \int_0^\infty f(x)e^{ix\zeta}\,dx$$

by dominated convergence; and similarly

$$\lim_{T\to\infty} \int_{ia-T}^{ia+T} \frac{F_+(w)K(w)}{w-\zeta}\,dw = \frac{1}{2\pi} \int_0^\infty f(x)\,dx \int_{-\infty}^\infty k(y)\,dy \lim_{T\to\infty} \int_{ia-T}^{ia+T} \frac{e^{iw(x+y)}}{w-\zeta}\,dw$$

$$= i \int_0^\infty f(x)\,dx \int_{-x}^\infty k(y)e^{i\zeta(x+y)}\,dy$$

$$= i \int_0^\infty e^{i\zeta t}\,dt \int_0^\infty f(x)k(t-x)\,dx.$$

Hence

$$\int_{ia-\infty}^{ia+\infty} \frac{F_+(w)\{1-\sqrt{(2\pi)}K(w)\}}{w-\zeta}\,dw$$

$$= \sqrt{(2\pi)}i \int_0^\infty e^{i\zeta t}\,dt \left\{f(t) - \int_0^\infty f(x)k(t-x)\,dx\right\}.$$

Similarly

$$\int_{ib-\infty}^{ib+\infty} \frac{F_-(w)\{1-\sqrt{(2\pi)}K(w)\}}{w-\zeta}\,dw = -\sqrt{(2\pi)}i \int_0^\infty e^{i\zeta t}\,dt \int_{-\infty}^0 f(x)k(t-x)\,dx.$$

Hence the sum of the terms on the left is zero, the result of Theorem 141 again holds, and the theorem follows as before.[†]

11.3. Examples. (i) Let

$$g(x) = e^{-|x|}, \qquad k(x) = \lambda e^x \ (x < 0), \quad 0 \ (x > 0).$$

Then

$$G(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^\infty e^{-|x|+ixu}\,dx = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{1+u^2},$$

$$K(u) = \frac{\lambda}{\sqrt{(2\pi)}} \int_{-\infty}^0 e^{x+ixu}\,dx = \frac{\lambda}{\sqrt{(2\pi)}} \frac{1}{1+iu}.$$

[†] A solution under different conditions is given by Bochner (2).

The L^2 solution is therefore

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{e^{-ixu}}{(1-iu)(1-\lambda+iu)}\,du.$$

Suppose, for example, that $0 < \lambda < 1$. Then

$$f(x) = \frac{2}{2-\lambda}e^{-x} \quad (x \geqslant 0), \qquad = \frac{2}{2-\lambda}e^{(1-\lambda)x} \quad (x < 0).$$

This is plainly a solution, and so the only L^2 solution. There are similar solutions for other values of λ.

The equation is

$$f(x) = e^{-|x|}+\lambda e^{x}\int\limits_{x}^{\infty} e^{-y}f(y)\,dy, \tag{11.3.1}$$

and is reducible to differential equations. Let

$$\phi(x) = \int\limits_{x}^{\infty} e^{-y}f(y)\,dy, \qquad \phi'(x) = -e^{-x}f(x).$$

Then, for $x > 0$, $\qquad -\phi'(x) = e^{-2x}+\lambda\phi(x),$

so that $\qquad \phi(x) = \dfrac{e^{-2x}}{2-\lambda}+Ce^{-\lambda x}.$

For $x < 0$, $\qquad -\phi'(x) = 1+\lambda\phi(x),$

so that $\qquad \phi(x) = -\dfrac{1}{\lambda}+C'e^{-\lambda x}.$

Since $\phi(x)$ is continuous at $x = 0$, $C' = C+\dfrac{1}{2-\lambda}+\dfrac{1}{\lambda}$. Hence

$$f(x) = \frac{2}{2-\lambda}e^{-x}+C\lambda e^{(1-\lambda)x} \quad (x > 0)$$

$$= \left(\frac{2}{2-\lambda}+C\lambda\right)e^{(1-\lambda)x} \quad (x < 0).$$

The complete solution therefore contains a term with an arbitrary constant; and in fact

$$f(x) = e^{(1-\lambda)x}$$

is a solution of the homogeneous equation

$$f(x) = \lambda\int\limits_{x}^{\infty} e^{x-y}f(y)\,dy, \tag{11.3.2}$$

corresponding to the zero $w = i(1-\lambda)$ of the function

$$1-\sqrt{(2\pi)}K(w) = 1-\frac{\lambda}{1+iw}.$$

(ii) Let† $k(x) = \lambda e^{-|x|}$ $(\lambda < \frac{1}{2})$. Then

$$K(u) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\lambda}{1+x^2}, \qquad M(u) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\lambda}{1+x^2-2\lambda},$$

$$m(x) = \frac{\lambda}{\sqrt{(1-2\lambda)}}e^{-|x|\sqrt{(1-2\lambda)}},$$

and, if $g(x)$ is L^2, the L^2 solution is

$$f(x) = g(x)+\frac{\lambda}{\sqrt{(1-2\lambda)}}\int\limits_{-\infty}^{\infty} g(t)e^{-|t-x|\sqrt{(1-2\lambda)}}\,dt.$$

Also
$$1-\sqrt{(2\pi)}K(w) = \frac{1+w^2-2\lambda}{1+w^2},$$

so that
$$Ae^{x\sqrt{(1-2\lambda)}}+Be^{-x\sqrt{(1-2\lambda)}}$$

is a solution of the homogeneous equation if $\lambda > 0$.

(iii) Consider the homogeneous equation in which $k(x) = e^{-\frac{1}{2}x^2}$.

(iv) Let $\lambda = \frac{1}{\pi}$, $k(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $g(x) = \frac{x}{1+x^2}$. Then

$$K(u) = \sqrt{\left(\frac{1}{2\pi}\right)}e^{-|u|}, \qquad G(u) = i\operatorname{sgn} u\,\pi e^{-|u|},$$

and the solution is

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty}\frac{i\pi\operatorname{sgn} u\, e^{-|u|}}{1-e^{-|u|}}e^{-ixu}\,du$$

$$= \sqrt{(2\pi)}\int\limits_{0}^{\infty}\frac{\sin xu}{e^u-1}\,du = \sqrt{(2\pi)}\left(\frac{\pi}{2}\coth\pi x-\frac{1}{2x}\right).$$

This just fails to come under the above conditions, and in fact $f(x)$ is not L^2.

(v) Let‡ $k(u) = \lambda/(1+u)$ in (11.1.6). Then $\mathfrak{K}(s) = \lambda\pi\operatorname{cosec} s\pi$, and the solution is

$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\mathfrak{G}(s)}{1-\lambda\pi\operatorname{cosec} s\pi}x^{-s}\,ds,$$

or
$$f(x) = g(x)+\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\lambda\pi}{\sin s\pi-\lambda\pi}\mathfrak{G}(s)x^{-s}\,ds.$$

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† Picard (1). ‡ A. C. Dixon (1).

If $\lambda\pi = \sin\alpha\pi$, where $0 < \alpha < \frac{1}{2}$, we have

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{\sin\alpha\pi}{\sin s\pi - \sin\alpha\pi} u^{-s}\,du = \frac{\tan\alpha\pi}{\pi}\,\frac{u^{-\alpha}-u^{1+\alpha}}{1-u^2},$$

and, by Parseval's formula, the solution may be written

$$f(x) = g(x)+x\frac{\tan\alpha\pi}{\pi}\int\limits_0^\infty g(y)\frac{(x/y)^{-1-\alpha}-(x/y)^\alpha}{y^2-x^2}\,dy$$

(vi) The homogeneous equation

$$f(x) = \lambda\int\limits_0^\infty \frac{f(y)}{x+y}\,dy \tag{11.3.3}$$

is reduced by the substitutions

$$x = e^\xi, \qquad y = e^\eta, \qquad e^{\frac{1}{2}\xi}f(e^\xi) = \phi(\xi)$$

to
$$\phi(\xi) = \lambda\int\limits_{-\infty}^\infty \frac{\phi(\eta)}{2\cosh\frac{1}{2}(\xi-\eta)}\,d\eta.$$

The only solutions of this of the form $\phi(\xi) = O(e^{c|\xi|})$, $0 < c < \frac{1}{2}$, are exponentials. We have

$$k(\xi) = \frac{\lambda}{2\cosh\frac{1}{2}\xi}, \qquad K(w) = \frac{\lambda\sqrt{\pi}}{\sqrt{2}\cosh\pi w},$$

and
$$1-\sqrt{(2\pi)}K(w) = 1-\frac{\pi\lambda}{\cosh\pi w}.$$

This has an infinity of zeros, some of which may lie in $-\frac{1}{2} < v < \frac{1}{2}$, and give solutions. For example, if $\lambda = 1/\pi$, there is a double zero at $w = 0$, and

$$\phi(\xi) = A+B\xi$$

is the solution, i.e.
$$f(x) = \frac{A+B\log x}{\sqrt{x}}$$

is the solution of (11.3.3).

That there are in fact no other solutions of any kind is proved by Hardy and Titchmarsh (3).

(vii) The homogeneous equation

$$x^\alpha f(x) = \frac{\lambda}{\Gamma(\alpha)}\int\limits_0^x (x-y)^{\alpha-1}f(y)\,dy \quad (0 < \alpha < 1) \tag{11.3.4}$$

is reduced by the substitution

$$x = e^\xi, \qquad y = e^\eta, \qquad e^{\alpha\xi}f(e^\xi) = \phi(\xi)$$

to the form
$$\phi(\xi) = \frac{\lambda}{\Gamma(\alpha)} \int_{-\infty}^{\xi} (e^{\xi-\eta}-1)^{\alpha-1}\phi(\eta)\,d\eta.$$

Here
$$k(\xi) = \frac{\lambda}{\Gamma(\alpha)}(e^{\xi}-1)^{\alpha-1} \quad (\xi \geqslant 0), \qquad 0 \quad (\xi < 0),$$

and
$$K(w) = \frac{\lambda}{\sqrt{(2\pi)}}\frac{\Gamma(1-iw-\alpha)}{\Gamma(1-iw)} \quad (v > -(1-\alpha)).$$

11.4. Various forms. Various other forms of equation are reducible to that just considered.

For example, consider†
$$f(x) = g(x)+\int_0^x \frac{1}{y}k\left(\frac{y}{x}\right)f(y)\,dy. \tag{11.4.1}$$

Putting $x = e^{-\xi}$, $y = e^{-\eta}$, and writing
$$f(e^{-\xi}) = \phi(\xi), \qquad g(e^{-\xi}) = \psi(\xi), \qquad k(e^{\xi}) = \kappa(\xi),$$

we obtain
$$\phi(\xi) = \psi(\xi)+\int_{\xi}^{\infty} \kappa(\xi-\eta)\phi(\eta)\,d\eta. \tag{11.4.2}$$

This is of the standard form if $\kappa(\xi) = 0$ for $\xi > 0$.

Another related form is
$$g(x) = \int_0^x k\left(\frac{y}{x}\right)f(y)\,dy. \tag{11.4.3}$$

If $f_1(x) = \int_0^x f(t)\,dt$, and k is an integral,
$$\int_0^x k\left(\frac{y}{x}\right)f(y)\,dy = k(1)f_1(x)-\frac{1}{x}\int_0^x k'\left(\frac{y}{x}\right)f_1(y)\,dy,$$

and if $k(1) \neq 0$ the equation is
$$f_1(x) = \frac{g(x)}{k(1)}+\int_0^x \frac{1}{y}\frac{1}{k(1)}\frac{y}{x}k'\left(\frac{y}{x}\right)f_1(y)\,dy.$$

This is of the same form as (11.4.1).

11.5. The equation with finite limits. Another equation of some interest is obtained by putting $f(x) = 0$, $g(x) = 0$, $k(x) = 0$, for $x < 0$, in (11.1.1). We obtain the equation
$$f(x) = g(x)+\int_0^x k(x-y)f(y)\,dy \quad (x > 0), \tag{11.5.1}$$

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† Browne (1).

considered by Doetsch[†] and Fock.[‡] Theorem 145 of course still applies; but now there is a more general solution of the same type.

THEOREM 147. *Let* $g(x)e^{-cx}$ *belong to* $L^2(0,\infty)$, *and* $k(x)e^{-cx}$ *to* $L(0,\infty)$, *for some positive c. Then there is just one solution* $f(x)$ *of* (11.5.1) *such that* $f(x)e^{-c'x}$ *belongs to* $L^2(0,\infty)$ *for some positive* c'; *it is given by*

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \operatorname*{l.i.m.}_{\lambda\to\infty} \int\limits_{ia-\lambda}^{ia+\lambda} \frac{G(w)}{1-\sqrt{(2\pi)}K(w)} e^{-ixw}\,dw \tag{11.5.2}$$

if a is sufficiently large.

The equation (11.5.1) is unchanged if we replace $f(x)$, $g(x)$, and $k(x)$ by $f(x)e^{-ax}$, $g(x)e^{-ax}$, and $k(x)e^{-ax}$ respectively, and we may argue in terms of these functions; or, what comes to the same thing, we may apply the argument of § 11.1 to $K(u+ia)$, etc., instead of to $K(u)$. We have

$$|K(u+ia)| \leqslant \frac{1}{\sqrt{(2\pi)}} \int\limits_0^\infty |k(x)|e^{-ax}\,dx < \frac{1}{\sqrt{(2\pi)}}$$

if a is sufficiently large. The solution then proceeds as before.

The solution (11.5.2) may also be written

$$f(x) = g(x) + \operatorname{l.i.m.} \int\limits_{ia-\lambda}^{ia+\lambda} G(w)\frac{K(w)}{1-\sqrt{(2\pi)}K(w)} e^{-ixw}\,dw. \tag{11.5.3}$$

Suppose that $k(x)e^{-cx}$ is also L^2. Then $K(w)$ is L^2, and hence so is

$$M(w) = \frac{K(w)}{1-\sqrt{(2\pi)}K(w)}, \tag{11.5.4}$$

and $$m(x) = \frac{1}{\sqrt{(2\pi)}} \operatorname{l.i.m.} \int\limits_{ia-\lambda}^{ia+\lambda} M(w)e^{-ixw}\,dw.$$

Making $a \to \infty$, it is seen that $m(x) = 0$ for $x < 0$. The solution can therefore be put in the form

$$f(x) = g(x) + \int\limits_0^x g(y)m(x-y)\,dy. \tag{11.5.5}$$

The relation (11.5.4) is equivalent to

$$m(x) = k(x) + \int\limits_0^x k(t)m(x-t)\,dt; \tag{11.5.6}$$

[†] Doetsch (1), (2).

[‡] Fock (1).

in fact it is at once verified that (11.5.5) is a solution of (11.5.1) if (11.5.6) holds and the inversions are justified.

(11.5.4) also gives

$$M(w) = \sum_{n=1}^{\infty} (2\pi)^{\frac{1}{2}n-\frac{1}{2}}\{K(w)\}^n,$$

and this is equivalent to

$$m(x) = \sum_{n=1}^{\infty} k^{(n)}(x),$$

where $\quad k^{(1)}(x) = k(x), \qquad k^{(n)}(x) = \int\limits_0^x k(t)k^{(n-1)}(x-t)\,dt.$

This is the well-known Volterra form of the solution.[†]

It has been proved[‡] by Wiener that, if $k(x)$ is $L(0,\infty)$, a necessary and sufficient condition that (11.5.6) should have a solution $h(x)$ of $L(0,\infty)$ is that $1-\sqrt{(2\pi)}K(u) \neq 0$ for u real. This is bound up with Wiener's Tauberian theory, which we do not discuss here.

Examples. (i) Let $k(x) = \lambda e^x$ $(x > 0)$, 0 $(x < 0)$. Then

$$\sqrt{(2\pi)}K(w) = -\frac{\lambda}{1+iw}, \qquad M(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{\lambda}{1+\lambda+iw},$$

$$m(x) = \lambda e^{x(1+\lambda)}.$$

Hence the solution of

$$f(x) = g(x)+\lambda\int\limits_0^x e^{x-y}f(y)\,dy$$

is

$$f(x) = g(x)+\lambda\int\limits_0^x e^{(1+\lambda)(x-y)}g(y)\,dy.$$

If $\phi(x) = \int\limits_0^x e^{-y}f(y)\,dy$, the equation reduces to the differential equation

$$\phi'(x)-\lambda\phi(x) = e^{-x}g(x).$$

This gives for $f(x)$ the above solution, together with $Ae^{(\lambda+1)x}$; but $A = 0$, since all the other terms vanish for $x < 0$.

(ii) Let[||] $k(x)$ be a finite sum of exponentials,

$$k(x) = Pe^{px}+Qe^{qx}+\dots \quad (x > 0).$$

Then

$$K(w) = -\frac{1}{\sqrt{(2\pi)}}\left\{\frac{P}{p+iw}+\frac{Q}{q+iw}+\dots\right\}.$$

[†] See Goursat's *Cours d'analyse*, t. 3, § 548–9.

[‡] Paley and Wiener, *Fourier Transforms*, § 18.

[||] E. T. Whittaker (1).

Hence $M(w)$ is a rational function, and may be written

$$M(w) = \frac{(p+iw)(q+iw)...}{(\alpha+iw)(\beta+iw)...}K(w),$$

where $i\alpha$, $i\beta$,... are the zeros of $1-\sqrt{(2\pi)}K(w)$. The calculus of residues then gives

$$m(x) = -\sum \frac{(p-\alpha)(q-\alpha)...}{(\beta-\alpha)(\gamma-\alpha)...}e^{\alpha x},$$

since $\sqrt{(2\pi)}K(i\alpha) = 1$.

A similar expression may be obtained for the solution if $k(x)$ is a polynomial.

(iii) Let† $g(x) = k(x) = \lambda J_0(x)$. Then

$$G(w) = K(w) = \frac{\lambda}{\sqrt{(2\pi)}}\int\limits_0^\infty J_0(x)e^{ixw}\,dx = \frac{\lambda}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(1-w^2)}},$$

and the solution is

$$f(x) = \frac{\lambda}{2\pi}\int\limits_{ia-\infty}^{ia+\infty}\frac{e^{-ixw}}{\sqrt{(1-w^2)}-\lambda}\,dw = \frac{\lambda}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{e^{xs}}{\sqrt{(1+s^2)}-\lambda}\,ds \quad (a>0)$$

$$= \frac{\lambda}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\sqrt{(1+s^2)}-s}{1-\lambda^2+s^2}e^{xs}\,ds +$$

$$+\frac{\lambda}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{s}{1-\lambda^2+s^2}e^{xs}\,ds+\frac{\lambda^2}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{e^{xs}}{1-\lambda^2+s^2}\,ds$$

$$= \frac{\lambda}{\sqrt{(1-\lambda^2)}}\int\limits_0^x \sin\{\sqrt{(1-\lambda^2)}(x-y)\}\frac{J_1(y)}{y}\,dy +$$

$$+\lambda\cos\{\sqrt{(1-\lambda^2)}x\}+\frac{\lambda^2}{\sqrt{(1-\lambda^2)}}\sin\{\sqrt{(1-\lambda^2)}x\},$$

by (7.13.2), (7.13.3), and (7.13.8).

11.6. Another type. Another integral equation which can be solved formally by means of Fourier integrals is

$$g(x) = \int\limits_{-\infty}^{\infty} k(x-y)f(y)\,dy. \tag{11.6.1}$$

This gives formally

$$G(u) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} e^{ixu}\,dx\int\limits_{-\infty}^{\infty} k(x-y)f(y)\,dy$$

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† Fock (1).

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y)\, dy \int_{-\infty}^{\infty} k(x-y) e^{ixu}\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y)\, dy \int_{-\infty}^{\infty} k(t) e^{i(t+y)u}\, dt$$

$$= \sqrt{(2\pi)} F(u) K(u). \qquad (11.6.2)$$

Hence the solution is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(u)}{K(u)} e^{-ixu}\, du. \qquad (11.6.3)$$

For this to be an actual solution $K(u)$ has to satisfy special conditions.

THEOREM 148. *Let $g(x)$ belong to $L^2(-\infty,\infty)$, and $k(x)$ to $L(-\infty,\infty)$. Then, in order that there should be a solution $f(x)$ of $L^2(-\infty,\infty)$, it is necessary and sufficient that $G(u)/K(u)$ should belong to $L^2(-\infty,\infty)$.*

Suppose that g, k, and f belong to the given L-classes, and (11.6.1) holds. Then (11.6.2) holds, by Theorem 65, p. 90, and F is L^2. Hence G/K is L^2.

Conversely, if G/K is L^2, then f, defined by (11.6.3), is L^2, and, by Theorem 65, the transform of the right-hand side of (11.6.1) is

$$\sqrt{(2\pi)} K(u) \frac{1}{\sqrt{(2\pi)}} \frac{G(u)}{K(u)} = G(u).$$

Hence (11.6.1) holds.

A similar equation soluble in terms of Mellin transforms is

$$g(x) = \int_{0}^{\infty} k(xy) f(y)\, dy. \qquad (11.6.4)$$

This gives formally

$$\mathfrak{G}(s) = \int_{0}^{\infty} x^{s-1}\, dx \int_{0}^{\infty} k(xy) f(y)\, dy$$

$$= \int_{0}^{\infty} f(y)\, dy \int_{0}^{\infty} k(xy) x^{s-1}\, dx$$

$$= \int_{0}^{\infty} f(y) y^{-s}\, dy \int_{0}^{\infty} k(u) u^{s-1}\, du$$

$$= \mathfrak{F}(1-s)\mathfrak{K}(s).$$

Hence
$$\mathfrak{F}(s) = \frac{\mathfrak{G}(1-s)}{\mathfrak{K}(1-s)},$$
and the solution is
$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{\mathfrak{G}(1-s)}{\mathfrak{K}(1-s)} x^{-s}\,ds. \tag{11.6.5}$$

11.7. Laplace's integral equation.

This is
$$g(x) = \int\limits_0^\infty f(y)e^{-xy}\,dy. \tag{11.7.1}$$
A formal solution
$$f(x) = \frac{1}{2\pi i}\int\limits_{\frac12-i\infty}^{\frac12+i\infty} \frac{\mathfrak{G}(1-s)}{\Gamma(1-s)} x^{-s}\,ds \tag{11.7.2}$$
is given by (11.6.5). The equation can, however, be solved directly by Fourier's integral formula. This gives
$$g(ix) = \int\limits_0^\infty f(y)e^{-ixy}\,dy,$$
and hence
$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g(iy)e^{ixy}\,dy \quad (x > 0), \tag{11.7.3}$$
the right-hand side being zero for $x < 0$.

If $g(x)$ is given originally for real x, the solution (11.7.3) involves an appeal to analytic continuation. The solution (11.7.2), with the usual definition of $\mathfrak{G}$, only involves explicitly $g(x)$ for real x; but it contains the factor $1/\Gamma(1-s)$, which is exponentially large at infinity, and it seems difficult to justify it except by an argument involving analytic continuation. In fact the equation (11.7.1) can only be satisfied if $g(x)$ has the values assumed on the real axis by an analytic function $g(z)$ regular for $x > 0$, so that some reference to the analytic character of $g(x)$ is almost inevitable.

We shall prove that *a necessary and sufficient condition that* (11.7.2) *should exist in the mean-square sense, and define a solution of* (11.7.1) *belonging to* $L^2(0,\infty)$, *is that* $g(x)$ *should have the values assumed on the real axis by an analytic function* $g(z)$, *regular for* $|\arg z| < \frac{1}{2}\pi$, *and such that*
$$\int\limits_0^\infty |g(re^{i\theta})|^2\,dr < K \tag{11.7.4}$$
for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

Suppose first that $f(x)$ satisfies the equation and belongs to $L^2(0,\infty)$.

Plainly $g(z)$ is regular for $R(z) > 0$, i.e. $|\arg z| < \frac{1}{2}\pi$. Now by Theorem 99, p. 131, we can write

$$f(u) = f_{(+)}(u)+f_{(-)}(u),$$

where $f_{(+)}(w)$ is regular for $\arg w > 0$, $f_{(-)}(w)$ for $\arg w < 0$. Then if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,

$$g(re^{i\theta}) = \int\limits_0^\infty e^{-re^{i\theta}u}f_{(+)}(u)\,du + \int\limits_0^\infty e^{-re^{i\theta}u}f_{(-)}(u)\,du.$$

In the first integral we can turn the line of integration through an angle $\frac{1}{2}\pi-\theta$, and in the second through $-\frac{1}{2}\pi-\theta$. We obtain

$$g(re^{i\theta}) = e^{i(\frac{1}{2}\pi-\theta)}\int\limits_0^\infty e^{-ir\rho}f_{(+)}(\rho e^{i(\frac{1}{2}\pi-\theta)})\,d\rho + e^{-i(\frac{1}{2}\pi+\theta)}\int\limits_0^\infty e^{ir\rho}f_{(-)}(\rho e^{-i(\frac{1}{2}\pi+\theta)})\,d\rho,$$

and since $f_{(+)}$ and $f_{(-)}$ belong to L^2 along every line $\arg w = \text{const}$, (11.7.4) follows.

Conversely, suppose that $g(z)$ satisfies the above condition. We have

$$\mathfrak{G}(1-s) = \int\limits_0^\infty g(x)x^{-s}\,dx.$$

If $t > 0$, we rotate the line of integration through $-\frac{1}{2}\pi$, giving,

$$\mathfrak{G}(1-s) = -i\int\limits_0^\infty g(-iy)(ye^{-\frac{1}{2}\pi i})^{-s}\,dy$$

$$= -ie^{\frac{1}{2}\pi i\sigma}e^{-\frac{1}{2}\pi t}\int\limits_0^\infty g(-iy)y^{-\sigma-it}\,dy.$$

For $\sigma = \frac{1}{2}$ this is $e^{-\frac{1}{2}\pi t}$ multiplied by a function of $L^2(0,\infty)$; a similar argument with $t < 0$ and a rotation through $\frac{1}{2}\pi$ shows that $\mathfrak{G}(1-s)$ is $e^{-\frac{1}{2}\pi|t|}$ multiplied by a function of $L^2(-\infty,\infty)$. Also

$$|1/\Gamma(1-s)| = O(e^{\frac{1}{2}\pi|t|}).$$

Hence the integral in (11.7.2) exists in the mean-square sense. That the $f(x)$ so defined satisfies the equation follows from Theorem 72.

Alternative forms of solution have been given by Widder (1), Paley and Wiener, § 13.

11.8. Stieltjes's integral equation. If we iterate the previous equation, i.e. put

$$g(x) = \int\limits_0^\infty f(y)e^{-xy}\,dy, \qquad h(x) = \int\limits_0^\infty g(y)e^{-xy}\,dy,$$

we obtain formally

$$h(x) = \int_0^\infty e^{-xy}\,dy \int_0^\infty f(u)e^{-yu}\,du$$

$$= \int_0^\infty f(u)\,du \int_0^\infty e^{-(x+u)y}\,dy$$

$$= \int_0^\infty \frac{f(u)}{x+u}\,du, \tag{11.8.1}$$

another integral equation of a similar type. This equation has been considered in connexion with Stieltjes's moment problem.[†]

Putting $x = e^{\xi}$, $y = e^{\eta}$, $e^{\frac{1}{2}\xi}h(e^{\xi}) = \psi(\xi)$, $e^{\frac{1}{2}\xi}f(e^{\xi}) = \phi(\xi)$, the equation becomes

$$\psi(\xi) = \int_{-\infty}^{\infty} \frac{\phi(\eta)}{2\cosh\frac{1}{2}(\xi-\eta)}\,d\eta. \tag{11.8.2}$$

This is of the form (11.6.1), with

$$k(\xi) = \tfrac{1}{2}\operatorname{sech}\tfrac{1}{2}\xi, \qquad K(u) = \sqrt{(\tfrac{1}{2}\pi)}\operatorname{sech}\pi u,$$

and the formal solution is

$$\phi(\xi) = \frac{1}{\pi\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \Psi(u)\cosh\pi u\, e^{-i\xi u}\,du$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \Psi(u)(e^{-i(\xi+i\pi)u}+e^{-i(\xi-i\pi)u})\,du$$

$$= \frac{1}{2\pi}\{\psi(\xi+i\pi)+\psi(\xi-i\pi)\}, \tag{11.8.3}$$

or

$$f(x) = \frac{i}{2\pi}\{h(xe^{i\pi})-h(xe^{-i\pi})\}. \tag{11.8.4}$$

An appeal to analytic continuation is again obviously involved.

We shall show that *a necessary and sufficient condition that* (11.8.3) *should define a solution of* (11.8.2) *belonging to* $L^2(-\infty,\infty)$ *is that* $\psi(z)$ *should be an analytic function, regular for* $-\pi < y < \pi$, *and that*

$$\int_{-\infty}^{\infty} |\psi(x+iy)|^2\,dy < K$$

for $-\pi < y < \pi$.

[†] See Hardy (7), Paley and Wiener, § 14.

As in § 5.4, the condition implies that there are limit-functions $\psi(x+i\pi)$ and $\psi(x-i\pi)$ belonging to $L^2(-\infty,\infty)$.

If ϕ is L^2 and ψ is defined by (11.8.2), then $\psi(z)$ is plainly analytic for $-\pi < y < \pi$; and by Theorem 64,

$$\psi(z) = \sqrt{(\tfrac{1}{2}\pi)} \int\limits_{-\infty}^{\infty} \frac{\Phi(u)}{\cosh \pi u} e^{-izu}\, du,$$

where Φ is the transform of ϕ. Hence

$$\int\limits_{-\infty}^{\infty} |\psi(x+iy)|^2\, dy = \pi^2 \int\limits_{-\infty}^{\infty} \frac{\{\Phi(u)\}^2}{\cosh^2 \pi u} e^{2yu}\, du$$

$$\leqslant \pi^2 \int\limits_{-\infty}^{\infty} \frac{\{\Phi(u)\}^2}{\cosh^2 \pi u} e^{2\pi|u|}\, du.$$

Hence the condition is necessary.

Conversely, if ψ is of the given form, then

$$e^{\pm\pi u}\Psi(u) = \frac{1}{\sqrt{(2\pi)}} \int\limits_{-\infty}^{\infty} \psi(\xi \pm i\pi) e^{i\xi u}\, du$$

belong to $L^2(0,\infty)$ and $L^2(-\infty,0)$ respectively. Hence $\Psi(u)\cosh \pi u$ belongs to $L^2(-\infty,\infty)$, and (11.8.3) defines a function ϕ of L^2. That it is a solution of (11.8.2) follows from Theorem 64.

In terms of the original functions, a necessary and sufficient condition for (11.8.1) to have a solution of L^2 is that $g(z) = g(re^{i\theta})$ should be analytic for $-\pi < \theta < \pi$, and that

$$\int\limits_{0}^{\infty} |g(re^{i\theta})|^2\, dr$$

should be bounded for $-\pi < \theta < \pi$.

That (11.8.4) is a solution of the original problem is easily verified, for the right-hand side of (11.8.1) is then

$$\frac{i}{2\pi} \int\limits_{0}^{\infty} \frac{h(ue^{i\pi})}{x+u}\, du - \frac{i}{2\pi} \int\limits_{0}^{\infty} \frac{h(ue^{-i\pi})}{x+u}\, du.$$

Rotating the line of integration of the second integral through 2π, and allowing for the residue at $u = xe^{i\pi}$, we see that this is equal to $h(x)$.

11.9. Stieltjes's moment problem.† A note on Stieltjes's moment problem itself may be inserted here. The problem is to determine $f(x)$ such that

$$\int_0^\infty x^n f(x)\,dx = c_n \quad (n = 0, 1, \ldots),$$

where $c_0, c_1, \ldots$ are given.

Suppose that $f(x)e^{k\sqrt{x}}$ is $L(0,\infty)$ for some positive k. Let

$$\phi(s) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n s^{2n}}{(2n)!} = \int_0^\infty f(x) \sum_{n=0}^{\infty} \frac{(-1)^n x^n s^{2n}}{(2n)!}\,dx$$

$$= \int_0^\infty f(x)\cos s\sqrt{x}\,dx = 2\int_0^\infty \xi f(\xi^2)\cos s\xi\,d\xi.$$

The inversion is justified by the convergence of

$$\int_0^\infty |f(x)| \sum_{n=0}^{\infty} \frac{x^n |s|^{2n}}{(2n)!}\,dx = \int_0^\infty f(x)\cosh|s|\sqrt{x}\,dx,$$

provided that $|s| < k$. The final integral, however, converges if $s = \sigma + it$, $-k < t < k$. Hence $\phi(s)$ is an analytic function, regular in this strip, and $\phi(s) \to 0$ as $\sigma \to \pm\infty$ in the strip. Also

$$\xi f(\xi^2) = \frac{1}{\pi} \lim_{\lambda\to\infty} \int_0^\lambda \left(1 - \frac{s}{\lambda}\right)\phi(s)\cos s\xi\,ds$$

for almost all ξ. Hence $f(x)$ is unique apart from sets of zero measure.

To show that this is actually a solution, we have, if $a > 0$,

$$\frac{1}{2\pi i} \int_{ia-\infty}^{ia+\infty} \frac{e^{-is\xi}}{s^{2n+1}}\,ds = \frac{(-1)^{n+1}\xi^{2n}}{(2n)!} \quad (\xi > 0), \qquad 0 \quad (\xi < 0).$$

Hence

$$\int_0^\infty x^n f(x)\,dx = 2\int_0^\infty \xi^{2n+1} f(\xi^2)\,d\xi$$

$$= \frac{(-1)^{n+1} 2n!}{\pi i} \int_{-\infty}^{\infty} |\xi| f(\xi^2)\,d\xi \int_{ia-\infty}^{ia+\infty} \frac{e^{-is\xi}}{s^{2n+1}}\,ds$$

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† See Hardy (7).

$$= \frac{(-1)^{n+1}2n!}{\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{ds}{s^{2n+1}} \int\limits_{-\infty}^{\infty} |\xi| f(\xi^2) e^{-is\xi}\, d\xi$$

$$= \frac{(-1)^{n+1}2n!}{\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{\phi(s)}{s^{2n+1}}\, ds.$$

The inversion is justified by absolute convergence if $n > 0$, and by the bounded convergence of the s-integral if $n = 0$; in the latter case the final integral is $\lim\limits_{\lambda\to\infty} \int\limits_{ia-\lambda}^{ia+\lambda}$. The result is, of course, a case of Parseval's formula.

Now

$$\int\limits_{-\infty}^{\infty} \frac{\phi(\sigma+ia)}{(\sigma+ia)^{2n+1}}\, d\sigma = \int\limits_{-\infty}^{\infty} \frac{\phi(-\sigma'+ia)}{(-\sigma'+ia)^{2n+1}}\, d\sigma' = -\int\limits_{-\infty}^{\infty} \frac{\phi(\sigma'-ia)}{(\sigma'-ia)^{2n+1}}\, d\sigma',$$

since ϕ is even. Hence

$$\frac{1}{\pi i} \int\limits_{ia-\infty}^{ia+\infty} \frac{\phi(s)}{s^{2n+1}}\, ds = \frac{1}{2\pi i}\left(\int\limits_{ia-\infty}^{ia+\infty} - \int\limits_{-ia-\infty}^{-ia+\infty} \right) \frac{\phi(s)}{s^{2n+1}}\, ds = \frac{(-1)^{n+1}c_n}{2n!}$$

by the theorem of residues. The desired result therefore follows.

The method, of course, does not show whether a particular set of c_n correspond to a function $f(x)$ of the class considered. For example, if $c_n = 1$ for every n, then $\phi(s) = \cos s$, which is not the transform of a function integrable in the ordinary sense. It is here that Stieltjes integrals become relevant.

If $c_n = 1/(n+1)$, then

$$\phi(s) = 2s^{-2}(s \sin s + \cos s - 1),$$

and

$$\xi f(\xi^2) = \frac{1}{\pi} \int\limits_{0}^{\infty} \left(\frac{2 \sin s}{s} - \frac{1-\cos s}{s^2} \right) \cos s\xi\, ds$$

$$= \xi \quad (0 < \xi < 1), \qquad 0 \quad (\xi > 1).$$

Since

$$\int\limits_{0}^{\infty} x^n e^{-x^\mu \cos\alpha} \sin(x^\mu \sin\alpha)\, dx = \frac{1}{\mu}\Gamma\left(\frac{n+1}{\mu}\right) \sin\frac{(n+1)\alpha}{\mu}$$

if $\mu > 0$ and $0 < \alpha < \frac{1}{2}\pi$, the function

$$f(x) = e^{-x^\mu \cos\mu\pi} \sin(x^\mu \sin\mu\pi)$$

satisfies
$$\int_0^\infty x^n f(x)\,dx = 0 \quad (n = 0, 1, \ldots)$$
for every value of μ less than $\frac{1}{2}$. The solution is therefore not unique if we merely assume that $f(x) = O(e^{-kx^\mu})$, where $\mu < \frac{1}{2}$.

11.10. Finite limits. The equation
$$g(x) = \int_0^x k(x-y)f(y)\,dy \quad (x > 0) \tag{11.10.1}$$
is, formally, the particular case of (11.6.1) in which $f(x)$ and $k(x)$, and so also $g(x)$, vanish for $x < 0$. The formal process of § 11.6 gives as before
$$G(w) = \sqrt{(2\pi)}F(w)K(w), \tag{11.10.2}$$
and the formal solution is
$$f(x) = \frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \frac{G(w)}{K(w)} e^{-ixw}\,dw. \tag{11.10.3}$$
As before, for this to be a solution, special conditions on $K(w)$ or special relations between $G(w)$ and $K(w)$ are required.

Theorem 149. *Let $g(x)e^{-cx}$ belong to $L^2(0,\infty)$, and $k(x)e^{-cx}$ to $L(0,\infty)$. Then, in order that there should be a solution $f(x)$ such that $f(x)e^{-cx}$ belongs to $L^2(0,\infty)$, it is necessary and sufficient that*
$$\int_{-\infty}^{\infty} \left|\frac{G(u+iv)}{K(u+iv)}\right|^2 du \leqslant M,$$
where M is a constant independent of v, for all $v \geqslant c$.

We can replace $f(x)$, $g(x)$, and $k(x)$ in (11.10.1) by $e^{-ax}f(x)$, $e^{-ax}g(x)$ and $e^{-ax}k(x)$ respectively, and the result follows from Theorems 148 and 95.

That the solution of (11.10.1), if it exists, is unique, can be proved more generally.

Theorem 150. *Let $f(x)e^{-cx}$ and $k(x)e^{-cx}$ belong to $L(0, \infty)$ for some positive c, and let*
$$\int_0^x k(x-y)f(y)\,dy = 0 \quad (x > 0).$$
Then at least one of k and f is null.

For (11.10.2) holds, the inversion in the proof being justified by absolute convergence; and now $G(w) = 0$. Hence either $F(w)$ or $K(w)$ is 0 for all w, and the corresponding f or k is null, by Theorem 14.

We shall next show that the same result holds without any restriction on the behaviour of the functions at infinity.† We use the following lemmas.

LEMMA α. *Let $\phi(w)$ be regular in the upper half-plane,*

$$\phi(w) = O(e^{k|w|}), \qquad |\phi(u)| \leqslant 1 \quad (w = u+iv),$$

and let $\phi(iv)$ be real. If $\epsilon > 0$, the connected region in which

$$|\phi(w)| \geqslant 1+\epsilon,$$

if it exists, contains arbitrarily large purely imaginary values of w.

Let w_1 be a point (if there is one) at which $|\phi(w_1)| \geqslant 1+\epsilon$, let D be the connected region containing w_1 in which $|\phi(w)| \geqslant 1+\epsilon$, and let D_1 and D_2 be the parts of D in the first and second quadrants. If the lemma is false, D_1 and D_2 meet at most along a finite stretch of the imaginary axis. Let $|\phi(w)| \leqslant m$ on this stretch. Since $\phi(w) = 1+\epsilon$ on the boundary of D, $|\phi(w)| \leqslant M = \max(1+\epsilon, m)$ on the boundary of D_1, and so, by the Phragmén-Lindelöf theorem, throughout D_1; and similarly throughout D_2.

But actually $m \leqslant 1+\epsilon$, so that $M = 1+\epsilon$. For the function $\psi(w) = (w+i)^{-\eta}\phi(w)$, where $\eta > 0$, satisfies $|\psi(w)| \leqslant 1+\epsilon$ on the boundary of D, and $\psi(w) \to 0$ as $|w| \to \infty$ in D. Hence $|\psi(w)| \leqslant 1+\epsilon$ throughout D. Hence

$$|\phi(w)| \leqslant (1+\epsilon)|w+i|^{\eta}$$

throughout D, and, making $\eta \to 0$, $|\phi(w)| \leqslant 1+\epsilon$. Since the reversed inequality also holds, $\phi(w) = C$, where $|C| = 1+\epsilon$. This is inconsistent with $|\phi(u)| \leqslant 1$, so that D must contain arbitrarily large purely imaginary values. Also since $\phi(iv)$ is real, the region

$$|\phi(1+w)| \geqslant 1+\epsilon$$

is symmetrical about the imaginary axis, and it is easily seen that two regions with the properties of the above D would have to overlap. Hence there is only one such connected region.

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† Titchmarsh (8), Crum (2).

LEMMA β. *Let $F(w)$ and $K(w)$ both have the properties of the above $\phi(w)$, and let $|F(w)K(w)| \leqslant e^{-\gamma v}$, where $\gamma > 0$, for all $v > 0$. Then there exist α and β such that $\alpha+\beta = \gamma$, and $|F(w)| \leqslant e^{-\alpha v}$, $|K(w)| \leqslant e^{-\beta v}$ for all $v > 0$.*

Consider the regions D and D' in which

$$|F(w)e^{\alpha' w}| \geqslant 1+\epsilon, \qquad |K(w)e^{\beta' w}| \geqslant 1+\epsilon,$$

where α' and β' are any fixed real numbers whose sum is γ, and $\epsilon > 0$. We shall show that either D or D' is empty.

By applying Lemma α to $F(w)e^{\alpha' w}$ and $K(w)e^{\beta' w}$, we see that D and D', if they exist, both contain arbitrarily large purely imaginary values of w. Let iv_1 be a point of D, iv_2 a point of D' with $v_2 > v_1$, and iv_3 a point of D with $v_3 > v_2$. Since D is a connected region, and is symmetrical about the imaginary axis, there must be a closed curve joining iv_1 and iv_3, lying entirely in D, surrounding iv_2. On this $|F(w)e^{\alpha' w}| \geqslant 1+\epsilon$, and so

$$|K(w)e^{(\gamma-\alpha')iw}| \leqslant 1/(1+\epsilon).$$

This inequality therefore holds throughout the area enclosed by the curve, and in particular at iv_2. This involves a contradiction, so that either D or D' is empty.

Suppose that, for some w_1 and w_2,

$$|F(w_1)| > e^{-\alpha' v_1} \quad \text{and} \quad |K(w_2)| > e^{-\beta' v_2}.$$

Then, for some positive ϵ,

$$|F(w_1)| \geqslant (1+\epsilon)e^{-\alpha' v_1}, \qquad |K(w_2)| \geqslant (1+\epsilon)e^{-\beta' v_2}.$$

Since we have shown that this cannot be so, it follows that either $|F(w)| \leqslant e^{-\alpha' v}$ for all $v > 0$, or $|K(w)| \leqslant e^{-\beta' v}$ for all $v > 0$. Let α be the upper bound of values of α' for which the first inequality holds. If it held for all α', $F(w)$ would be identically zero; if it held for no α', the second inequality would hold for all β', and $K(w)$ would be identically zero. Otherwise $0 < \alpha < \infty$, $|F(w)| \leqslant e^{-(\alpha-\epsilon)v}$ for all w and arbitrarily small ϵ, and so $|F(w)| \leqslant e^{-\alpha v}$. If $\alpha' = \alpha+\epsilon$, the second of the above alternatives holds, so that

$$|K(w)| \leqslant e^{-(\gamma-\alpha-\epsilon)v} = e^{-(\beta-\epsilon)v}$$

for all w, and so $|K(w)| \leqslant e^{-\beta v}$ for all w. This proves the lemma.

THEOREM 151. *Let f and k belong to $L(0, \gamma)$, and let*

$$g(x) = \int_0^x f(y)k(x-y)\,dy = 0$$

for almost all x in $(0,\gamma)$. Then $f(x) = 0$ for almost all x in $(0,\alpha)$, and $k(x) = 0$ for almost all x in $(0,\beta)$, where $\alpha+\beta = \gamma$.

We may suppose that

$$\frac{1}{\sqrt{(2\pi)}}\int\limits_0^\gamma |f(x)|\,dx \leqslant 1, \qquad \frac{1}{\sqrt{(2\pi)}}\int\limits_0^\gamma |k(x)|\,dx \leqslant 1,$$

and that $f(x)$ and $k(x)$ are null for $x < 0$ and $x > \gamma$. Then $g(x)$ is null for $x < \gamma$ and $x > 2\gamma$, and

$$\begin{aligned}\int\limits_\gamma^{2\gamma} |g(x)|\,dx &\leqslant \int\limits_0^{2\gamma} dx \int\limits_0^x |f(y)k(x-y)|\,dy \\ &= \int\limits_0^{2\gamma} |f(y)|\,dy \int\limits_y^{2\gamma} |k(x-y)|\,dx \\ &\leqslant \int\limits_0^{2\gamma} |f(y)|\,dy \int\limits_0^{2\gamma} |k(t)|\,dt \leqslant 2\pi.\end{aligned}$$

As before, the transforms $F(w)$ of $f(x)$, etc., are related by

$$G(w) = \sqrt{(2\pi)}F(w)K(w);$$

hence

$$|F(w)K(w)| = \frac{1}{2\pi}\left|\int\limits_\gamma^{2\gamma} g(x)e^{ixw}\,dx\right| \leqslant \frac{e^{-\gamma v}}{2\pi}\int\limits_\gamma^{2\gamma} |g(x)|\,dx \leqslant e^{-\gamma v}.$$

Hence, by Lemma β, either $F(w) \equiv 0$ or $K(w) \equiv 0$, or there exist α and β such that $\alpha+\beta = \gamma$, and $|F(w)| \leqslant e^{-\alpha v}$, $|K(w)| \leqslant e^{-\beta v}$ for $v > 0$. Now

$$\int\limits_0^\xi f(x)\,dx = \frac{1}{\sqrt{(2\pi)}}\lim_{\lambda\to\infty}\int\limits_{-\lambda}^{\lambda} F(w)\frac{e^{-i\xi w}-1}{-iw}\,dw$$

by Theorem 22, and the ordinary method of integrating round a contour in the upper half-plane shows that this is 0 if $\xi < \alpha$. Hence $f(x)$ is null in $(0,\alpha)$, and similarly $k(x)$ is null in $(0,\beta)$.

THEOREM 152. *If f and k are integrable over any finite interval, k is not null, and*

$$\int\limits_0^x f(y)k(x-y)\,dy = 0 \quad (0 < x < \infty),$$

then f is null in $(0,\infty)$.

By the previous theorem f is null in $(0,\alpha)$, where $\alpha+\beta = \gamma$, γ is arbitrarily large, β bounded.

11.11. Another example of an equation with finite limits is[†]

$$f(x) = \int_0^1 k(t)f(x-t)\,dt, \tag{11.11.1}$$

where $k(t)$ belongs to $L^2(0,1)$, and $f(t)$ to L^2 over any finite interval. Here the integral represents a continuous function, so that $f(x)$ is, in fact, continuous.

Let $f(x) = e^{cx}g(x)$. Then

$$g(x) = \int_0^1 k(t)e^{-ct}g(x-t)\,dt.$$

Taking c so large that $\int_0^1 |k(t)|e^{-ct}\,dt < 1$, it follows that

$$|g(x)| < \max_{x-1\leqslant\xi\leqslant x} |g(\xi)|,$$

and hence that $g(x)$ is bounded as $x \to \infty$. Hence $f(x) = O(e^{cx})$. If we assume also that $f(x) = O(e^{c|x|})$ as $x \to -\infty$, the theory of § 11.2 applies, (11.11.1) being the particular case of (11.2.1) in which $k(t) = 0$ for $t < 0$ and for $t > 1$.

We can, however, prove without this assumption

Theorem 153. *The solution of* (11.11.1) *is*

$$f(x) = \sum C_\nu e^{-iw_\nu x}, \tag{11.11.2}$$

where w_ν runs through the zeros of

$$G(w) = 1 - \int_0^1 k(t)e^{iwt}\,dt$$

with $\mathbf{I}(w_\nu) \leqslant c$, *and C_ν is a constant at simple zeros, linear at double zeros, etc.*

Let
$$F_\alpha(w) = \frac{1}{\sqrt{(2\pi)}}\int_\alpha^\infty f(x)e^{iwx}\,dx \tag{11.11.3}$$

(cf. § 10.16). Then $F_\alpha(w)$ is regular for $v > c$, with the above c. The formal argument is then as follows. If $a > c$,

$$\frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} F_\alpha(w)e^{-iwx}\,dw = \begin{cases} f(x) & (x > \alpha), \\ 0 & (x < \alpha). \end{cases} \tag{11.11.4}$$

[†] Schurer (1), Titchmarsh (16).

Hence, if $x > \alpha+1$,

$$\int\limits_0^1 k(t)f(x-t)\,dt = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_\alpha(w)e^{-iwx}\,dw \int\limits_0^1 k(t)e^{iwt}\,dt,$$

and (11.11.1) gives

$$\int\limits_{ia-\infty}^{ia+\infty} F_\alpha(w)G(w)e^{-iwx}\,dw = 0 \quad (x > \alpha+1).$$

Multiplying by $e^{ix\zeta}$, where $\zeta = \xi+i\eta$, $\eta > a$, and integrating over $(\alpha+1,\infty)$, we obtain

$$\int\limits_{ia-\infty}^{ia+\infty} F_\alpha(w)G(w)\frac{e^{i(\zeta-w)(\alpha+1)}}{\zeta-w}\,dw = 0.$$

The result may be justified by mean-square theory, as in Theorem 141.

Moving the integral to the parallel line through ia', where $a' > \eta$, we obtain

$$\int\limits_{ia'-\infty}^{ia'+\infty} F_\alpha(w)G(w)\frac{e^{i(\zeta-w)(\alpha+1)}}{\zeta-w}\,dw = -2\pi i F_\alpha(\zeta)G(\zeta).$$

The left-hand side is an analytic function of ζ, regular for $\eta < a'$. It therefore provides the analytic continuation of the right-hand side throughout $\eta < a'$. It follows that $F_\alpha(\zeta)$ is regular for $\eta < a'$, except possibly for poles at the zeros of $G(\zeta)$. Also

$$F_\alpha(\zeta)G(\zeta) = o\left(e^{-\eta(\alpha+1)}\right)$$

as $\zeta \to \infty$ uniformly for $\eta \leqslant a < a'$. If the zeros of $G(\zeta)$ are separated by suitable contours on which $|G(\zeta)| >$ const., (11.11.2) follows on applying the usual contour integration to (11.11.4). The result certainly holds if $k(t)$ is absolutely continuous near $t = 1$, and $k(1) \neq 0$; for then we can integrate by parts and obtain

$$G(w) = 1-\frac{k(1)e^{iw}}{iw}+o\left(\frac{e^{-v}}{|w|}\right)$$

from which the result easily follows.

Finally, the C_ν are independent of α; for example, at a simple zero of $G(w)$,

$$C_\nu = \frac{1}{\sqrt{(2\pi)}G'(w_\nu)} \int\limits_{ia'-\infty}^{ia'+\infty} F_\alpha(w)G(w)\frac{e^{i(w_\nu-w)(\alpha+1)}}{w_\nu-w}\,dw,$$

$$\frac{\partial C_\nu}{\partial\alpha} = -\frac{f(\alpha)e^{iw_\nu(\alpha+1)}}{2\pi G'(w_\nu)} \int\limits_{ia'-\infty}^{ia'+\infty} \frac{G(w)e^{-iw}}{w_\nu-w}\,dw +$$

$$+\frac{ie^{iw_\nu(\alpha+1)}}{\sqrt{(2\pi)}G'(w_\nu)} \int\limits_{ia'-\infty}^{ia'+\infty} F_\alpha(w)G(w)e^{-iw(\alpha+1)}\,dw.$$

Each of these integrals is zero by Cauchy's theorem, the integrands being regular in the half-plane below the path of integration.

Similar methods can be applied to the solution of many other problems.†

11.12. Examples. The following example of (11.10.1) is considered by Bateman (6). A tradesman buys and sells various articles. It is assumed (i) that buying and selling are continuous processes, and that goods bought begin to be sold at once; (ii) that when the tradesman buys a new supply of any article, he buys just as much as he can sell in time T, the same for all such purchases; (iii) that the new supply sells uniformly during the time T.

The tradesman starts with a new supply of unit value, and it is required to find the law according to which purchases must be made if the value of the stock is to remain constant.

The amount of the original stock remaining after time t is $k(t)$, where

$$k(t) = 1-t/T \quad (t \leqslant T), \qquad 0 \quad (t > T).$$

Suppose that articles of value $f(\tau)\,\delta\tau$ are purchased in the interval of time between τ and $\tau+\delta\tau$. This stock is reduced by sales in such a way that the value of the remainder at time $t > \tau$ is

$$k(t-\tau)f(\tau)\,\delta\tau.$$

The value at time t of the unsold stock due to purchases is therefore

$$\int\limits_0^t k(t-\tau)f(\tau)\,d\tau.$$

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† Busbridge (6), Cooper (2).

Hence f satisfies the integral equation

$$1-k(t) = \int_0^t k(t-\tau)f(\tau)\,d\tau.$$

Here

$$K(w) = \frac{1}{\sqrt{(2\pi)}}\int_0^T \left(1-\frac{t}{T}\right)e^{itw}\,dt = \frac{1}{\sqrt{(2\pi)}}\left(-\frac{1}{iw}+\frac{1-e^{iwT}}{Tw^2}\right),$$

and

$$G(w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty e^{itw}\,dt-K(w)$$

$$= -\frac{1}{\sqrt{(2\pi)}iw}-K(w) = -\frac{1}{\sqrt{(2\pi)}}\frac{1-e^{iwT}}{Tw^2}.$$

We can take $\mathbf{I}(w) > 0$, and the solution is

$$f(t) = -\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \frac{1-e^{iwT}}{iwT+1-e^{iwT}}e^{-iwt}\,dw \quad (a > 0).$$

This can be expanded in various forms. If we move the line of integration to a parallel line through $w = -ib$, where $b > 0$, we obtain

$$f(t) = \frac{2}{T}-\frac{1}{2\pi}\int_{-ib-\infty}^{-ib+\infty} \frac{1-e^{iwt}}{iwT+1-e^{iwT}}e^{-iwt}\,dw,$$

and the last integral is exponentially small as $t \to \infty$. Further terms in the approximation arise from the zeros of the denominator.†

11.13. As another example we shall sum the series‡

$$f(x) = \sum_{n=1}^{\infty} nJ_n(x)J_n(\xi). \tag{11.13.1}$$

We have $|J_n(x)| \leqslant 1$ for all n and x, and hence

$$|J_n(x)| = \left|\frac{x}{2n}\{J_{n-1}(x)+J_{n+1}(x)\}\right| \leqslant x$$

for $n \geqslant 1$ and $x \geqslant 0$. Also, for a fixed ξ, as $n \to \infty$

$$J_n(\xi) = O\{(\tfrac{1}{2}\xi)^n/n!\}.$$

Hence we may multiply (11.13.1) by $J_0(t-x)/x$ and integrate term-by-term over $(0,t)$. We obtain

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† See also Goldstein (1).

‡ See Watson, § 16.32.

$$\int\limits_0^t \frac{f(x)}{x} J_0(t-x)\,dx = \sum_{n=1}^{\infty} nJ_n(\xi) \int\limits_0^t \frac{J_n(x)}{x} J_0(t-x)\,dx$$

$$= \sum_{n=1}^{\infty} J_n(\xi)J_n(t) = \tfrac{1}{2}J_0(t-\xi) - \tfrac{1}{2}J_0(\xi)J_0(t),$$

by (7.14.6) and the 'addition formula' for Bessel coefficients.†

This is an integral equation for $f(x)/x$ of the form (11.10.1); by the above inequalities, $f(x)/x$ is bounded, so that it is given by (11.10.3). Here

$$K(w) = \frac{1}{\sqrt{(2\pi)}} \int\limits_0^{\infty} J_0(x)e^{ixw}\,dx = \frac{1}{\sqrt{\{2\pi(1-w^2)\}}},$$

where $u > 0$, and the branch which is real and positive on the real axis is taken. Similarly,

$$G(w) = \frac{1}{2\sqrt{(2\pi)}} \int\limits_0^{\infty} \{J_0(x-\xi) - J_0(x)J_0(\xi)\}e^{ixw}\,dx$$

$$= \frac{1}{2\sqrt{(2\pi)}} \frac{1}{iw} \int\limits_0^{\infty} \{J_1(x-\xi) - J_1(x)J_0(\xi)\}e^{ixw}\,dx$$

on integrating by parts. Hence

$$\frac{f(x)}{x} = \frac{1}{4\pi} \lim_{\lambda\to\infty} \int\limits_{ia-\lambda}^{ia+\lambda} \frac{\sqrt{(1-w^2)}}{iw} e^{-ixw}\,dw \int\limits_0^{\infty} \{J_1(t-\xi) - J_1(t)J_0(\xi)\}e^{itw}\,dt,$$

where $a > 0$. Now

$$-\frac{1}{4\pi} \lim_{\lambda\to\infty} \int\limits_{ia-\lambda}^{ia+\lambda} e^{-ixw}\,dw \int\limits_0^{\infty} \{J_1(t-\xi) - J_1(t)J_0(\xi)\}e^{itw}\,dt$$

$$= -\tfrac{1}{2}\{J_1(x-\xi) - J_1(x)J_0(\xi)\},$$

and the remainder is

$$\frac{1}{4\pi} \int\limits_{ia-\infty}^{ia+\infty} \frac{\sqrt{(1-w^2)}+iw}{iw} e^{-ixw}\,dw \int\limits_0^{\infty} \{J_1(t-\xi) - J_1(t)J_0(\xi)\}e^{itw}\,dt$$

$$= \frac{1}{4\pi} \int\limits_0^{\infty} \{J_1(t-\xi) - J_1(t)J_0(\xi)\}\,dt \int\limits_{ia-\infty}^{ia+\infty} \frac{\sqrt{(1-w^2)}+iw}{iw} e^{i(t-x)w}\,dw.$$

The inner integral is 0 if $t \geqslant x$ (by making $a \to \infty$). For $t < x$ its derivative with respect to t is $2\pi J_1(t-x)/(t-x)$, by (7.13.8). Hence, on integrating the repeated integral by parts, we obtain

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† See Watson, § 2.4.

$$\frac{1}{2}\int_0^x \{J_0(t-\xi)-J_0(t)J_0(\xi)\}\frac{J_1(t-x)}{t-x}\,dt$$

$$=\frac{1}{2}\int_0^x J_0(t-\xi)\frac{J_1(t-x)}{t-x}\,dt-\tfrac{1}{2}J_1(x)J_0(\xi),$$

by (7.14.6). Hence

$$\frac{f(x)}{x}=\frac{1}{2}\int_0^x J_0(t-\xi)\frac{J_1(t-x)}{t-x}\,dt-\tfrac{1}{2}J_1(x-\xi)=\frac{1}{2}\int_0^\xi J_0(t-\xi)\frac{J_1(t-x)}{t-x}\,dt,$$

again by (7.14.6).

11.14. Abel's integral equation.

This is

$$g(x)=\int_0^x (x-y)^{-\alpha}f(y)\,dy \quad (0<\alpha<1),$$

and is of the form (11.10.1), with $k(x)=x^{-\alpha}$. Here

$$K(w)=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty x^{-\alpha}e^{ixw}\,dx=\frac{1}{\sqrt{(2\pi)}}(-iw)^{\alpha-1}\Gamma(1-\alpha),$$

where $(-iw)^{\alpha-1}$ is real on the positive imaginary axis. The formal solution is therefore

$$f(x)=\frac{1}{\sqrt{(2\pi)}\Gamma(1-\alpha)}\int_{ia-\infty}^{ia+\infty}\frac{G(w)}{(-iw)^{\alpha-1}}e^{-ixw}\,dw.$$

If this is an L^2 solution, its integral is

$$f_1(x)=\frac{1}{\sqrt{(2\pi)}\Gamma(1-\alpha)}\int_{ia-\infty}^{ia+\infty}\frac{G(w)}{(-iw)^{\alpha-1}}\frac{1-e^{-ixw}}{iw}\,dw$$

$$=\frac{1}{2\pi\Gamma(1-\alpha)}\int_{ia-\infty}^{ia+\infty}\frac{e^{-ixw}-1}{(-iw)^{\alpha}}\,dw\int_0^\infty g(t)e^{iwt}\,dt$$

$$=\frac{1}{2\pi\Gamma(1-\alpha)}\int_0^\infty g(t)\,dt\int_{ia-\infty}^{ia+\infty}\frac{e^{iw(t-x)}-e^{iwt}}{(-iw)^{\alpha}}\,dw.$$

The inner integral is 0 if $t>x$ (by making $a\to+\infty$). For $0<t<x$ the contribution of e^{iwt} is still 0, while the other part is (by deforming

the line of integration into the negative imaginary axis)

$$\int_0^\infty v^{-\alpha}e^{v(t-x)}(e^{i\pi\alpha}-e^{-i\pi\alpha})(-i)\,dv = 2\sin\pi\alpha\,\Gamma(1-\alpha)(x-t)^{\alpha-1}.$$

Hence
$$f_1(x) = \frac{\sin\pi\alpha}{\pi}\int_0^x (x-t)^{\alpha-1}g(t)\,dt,$$

and
$$f(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x (x-t)^{\alpha-1}g(t)\,dt,$$

the usual form of solution.†

11.15. An equation of Fox. Another equation of 'resultant' type is

$$f(x) = g(x)+\int_{-\infty}^{\infty} k(x+y)f(y)\,dy. \tag{11.15.1}$$

This is equivalent to the equation considered by Fox (2). The solution is a little more elaborate. We have as before

$$F(u) = G(u)+\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ixu}\,dx\int_{-\infty}^{\infty} k(x+y)f(y)\,dy$$

$$= G(u)+\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy\int_{-\infty}^{\infty} k(x+y)e^{ixu}\,dx$$

$$= G(u)+\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy\int_{-\infty}^{\infty} k(t)e^{iu(t-y)}\,dt$$

$$= G(u)+\sqrt{(2\pi)}F(-u)K(u).$$

Changing the sign of u,

$$F(-u) = G(-u)+\sqrt{(2\pi)}F(u)K(-u),$$

and, eliminating $F(-u)$,

$$F(u) = \frac{G(u)+\sqrt{(2\pi)}G(-u)K(u)}{1-2\pi K(u)K(-u)}.$$

Hence
$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{G(u)+\sqrt{(2\pi)}G(-u)K(u)}{1-2\pi K(u)K(-u)}e^{-ixu}\,du. \tag{11.15.2}$$

The form actually considered by Fox is

$$f(x) = g(x)+\int_0^\infty k(ux)f(u)\,du, \tag{11.15.3}$$

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† See Bosanquet (1) for a direct study of the solution.

which is connected with (11.15.1) by obvious transformations. The corresponding analysis for this equation goes in terms of Mellin transforms, and the solution is

$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{\mathfrak{G}(s)+\mathfrak{K}(s)\mathfrak{G}(1-s)}{1-\mathfrak{K}(s)\mathfrak{K}(1-s)} x^{-s}\, ds. \tag{11.15.4}$$

Theorem 154. *Let $g(x)$ belong to L^2 and $k(x)$ to L, and let the upper bound of $K(u)K(-u)$ be less than $1/2\pi$. Then the equation* (11.15.1) *has just one solution of L^2, given by* (11.15.2).

As in § 11.1, $f(x)$ belongs to L^2 and satisfies the equation. Also the difference between two solutions of L^2 satisfies

$$f(x) = \int\limits_{-\infty}^{\infty} k(x+y)f(y)\, dy, \tag{11.15.5}$$

and so its transform satisfies

$$F(u) = \surd(2\pi)F(-u)K(u). \tag{11.15.6}$$

Hence $$F(-u) = \surd(2\pi)F(u)K(-u),$$

and so $$F(u)F(-u)\{1-2\pi K(u)K(-u)\} = 0.$$

Hence $F(u)$ or $F(-u)$ is 0 for almost all u. But, by (11.15.6), if $F(-u) = 0$, then $F(u) = 0$. Hence $F(u) = 0$ for almost all u, and hence $f(x) = 0$ for almost all x.

There are obvious extensions, e.g. we could simply say that $|1-2\pi K(u)K(-u)| \geqslant A > 0$.

Examples. (i) In (11.15.3) let

$$k(x) = \lambda\sqrt{\left(\frac{2}{\pi}\right)}\cos x.$$

Then $$\mathfrak{K}(s) = \lambda\Gamma(s)\cos\tfrac{1}{2}s\pi,$$

and $$\mathfrak{K}(s)\mathfrak{K}(1-s) = \lambda^2.$$

Hence, if $\lambda^2 \neq 1$, the solution is

$$\begin{aligned} f(x) &= \frac{1}{2\pi i(1-\lambda^2)}\int\limits_{c-i\infty}^{c+i\infty} \{\mathfrak{G}(s)+\lambda\mathfrak{G}(1-s)\Gamma(s)\cos\tfrac{1}{2}s\pi\}x^{-s}\, ds \\ &= \frac{1}{1-\lambda^2}g(x)+\frac{\lambda}{1-\lambda^2}\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{0}^{\infty} g(u)\cos xu\, du. \end{aligned}$$

This may be verified by Fourier's cosine formula.

(ii) In (11.15.3) let $k(x) = \pi^{-\frac{1}{2}}e^{-x}$ and

$$g(x) = \frac{\log(1+x)}{x} \quad (0 < x < 1), \qquad \frac{\log(1+x)}{x}-\frac{\pi}{x} \quad (x > 1).$$

Then $\qquad \mathfrak{K}(s) = \pi^{-\frac{1}{2}}\Gamma(s), \qquad \mathfrak{G}(s) = \frac{\pi}{1-s}\left(\frac{1}{\sin s\pi}-1\right),$

and the solution is

$$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\left\{\frac{\pi}{s-1}-\sqrt{\pi}\,\Gamma(s-1)\right\}x^{-s}\,ds$$

$$= \sqrt{\pi}\frac{1-e^{-x}}{x} \quad (0 < x < 1), \qquad \sqrt{\pi}\frac{1-e^{-x}}{x}-\frac{\pi}{x} \quad (x > 1).$$

11.16. 'Dual' integral equations. In some problems the unknown function satisfies one integral equation over part of the range $(0,\infty)$, and a different equation over the rest of the range.

For example,† let $v(\rho, z)$ be the potential of a flat circular electrified disk of conducting material, its centre being at the origin, and its axis along the axis of z. The potential satisfies the differential equation

$$\frac{\partial^2 v}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial\rho}+\frac{\partial^2 v}{\partial z^2} = 0. \tag{11.16.1}$$

Let $$V(u,z) = \int\limits_0^\infty \rho v(\rho,z)J_0(\rho u)\,d\rho \quad (z > 0). \tag{11.16.2}$$

Then $$\frac{\partial^2 V}{\partial z^2} = \int\limits_0^\infty \rho\frac{\partial^2 v}{\partial z^2}J_0(\rho u)\,du = -\int\limits_0^\infty \left(\rho\frac{\partial^2 v}{\partial\rho^2}+\frac{\partial v}{\partial\rho}\right)J_0(\rho u)\,d\rho,$$

and $$\int\limits_0^\infty \rho\frac{\partial^2 v}{\partial\rho^2}J_0(\rho u)\,d\rho = -\int\limits_0^\infty \frac{\partial v}{\partial\rho}\{J_0(\rho u)+\rho uJ_0'(\rho u)\}\,d\rho.$$

Hence

$$\frac{\partial^2 V}{\partial z^2} = \int\limits_0^\infty \frac{\partial v}{\partial\rho}\rho uJ_0'(\rho u)\,d\rho = -u\int\limits_0^\infty v\{J_0'(\rho u)+\rho uJ_0''(\rho u)\}\,d\rho$$

$$= u^2\int\limits_0^\infty v\rho J_0(\rho u)\,d\rho = u^2V.$$

Hence $$V = A(u)e^{-uz}+B(u)e^{uz},$$

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† Riemann-Weber, 1, § 134.

and plainly $B(u) = 0$. Hence, by Hankel's theorem,

$$v(\rho, z) = \int_0^\infty uA(u)e^{-uz}J_0(\rho u)\,du.$$

Taking the radius of the disk to be unity, the boundary conditions are

$$v = \text{const.} \quad (z = 0,\ 0 < \rho < 1); \qquad \frac{\partial v}{dz} = 0 \quad (z = 0,\ \rho > 1).$$

Hence, writing $uA(u) = f(u)$, $f(u)$ must satisfy

$$\int_0^\infty f(u)J_0(\rho u)\,du = g(\rho) \quad (0 < \rho < 1), \tag{11.16.3}$$

$$\int_0^\infty f(u)uJ_0(\rho u)\,du = 0 \qquad (\rho > 1), \tag{11.16.4}$$

where, in the above case, $g(\rho)$ is a constant.

To solve these equations formally†, apply Parseval's formula for Mellin transforms to the left-hand sides. We obtain

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\frac{2^{-s}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\rho^{s-1}\,ds = g(\rho) \quad (0 < \rho < 1), \tag{11.16.5}$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\frac{2^{1-s}\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\rho^{s-2}\,ds = 0 \qquad (\rho > 1), \tag{11.16.6}$$

where $0 < k < 1$. Putting

$$\mathfrak{F}(s) = \frac{2^s\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\chi(s), \tag{11.16.7}$$

the equations become

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\chi(s)\rho^{s-1}\,ds = g(\rho) \quad (0 < \rho < 1), \tag{11.16.8}$$

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\chi(s)\rho^{s-1}\,ds = 0 \qquad (\rho > 1). \tag{11.16.9}$$

In this form the Γ-function factor in (11.16.8) has no poles or zeros for $\sigma > 0$, and that in (11.16.9) has no poles or zeros for $\sigma < 1$.

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† See also Busbridge (2), Copson (2).

Multiplying (11.16.8) by ρ^{-w}, where $\sigma-u > 0$, and integrating over (0, 1), we obtain

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\frac{\chi(s)}{s-w}\,ds = \int\limits_0^1 g(\rho)\rho^{-w}\,d\rho = \mathfrak{G}(1-w) \quad (u < k).$$

Moving the line of integration to $\sigma = k' < u$,

$$\frac{1}{2\pi i}\int\limits_{k'-i\infty}^{k'+i\infty}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\frac{\chi(s)}{s-w}\,ds = \mathfrak{G}(1-w)-\frac{\Gamma(\frac{1}{2}w)}{\Gamma(\frac{1}{2}+\frac{1}{2}w)}\chi(w).$$

The left-hand side is regular for $u > k'$, and so for $u > 0$. Hence so is the right-hand side. Hence so also is

$$\chi(w)-\frac{\Gamma(\frac{1}{2}+\frac{1}{2}w)}{\Gamma(\frac{1}{2}w)}\mathfrak{G}(1-w).$$

Hence (assuming suitable conditions at infinity)

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\left\{\chi(s)-\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\mathfrak{G}(1-s)\right\}\frac{ds}{s-w} = 0 \quad (u < k). \tag{11.16.10}$$

Similarly, multiplying (11.16.9) by ρ^{-w}, where $\sigma-u < 0$, and integrating over $(1,\infty)$, we obtain

$$\frac{1}{2\pi i}\int\limits_{k'-i\infty}^{k'+i\infty}\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\frac{\chi(s)}{s-w}\,ds = 0 \quad (u > k').$$

We conclude as before that $\{\Gamma(1-\frac{1}{2}s)/\Gamma(\frac{1}{2}-\frac{1}{2}s)\}\chi(s)$, and so $\chi(s)$, is regular for $\sigma < 1$. Hence

$$\frac{1}{2\pi i}\int\limits_{k'-i\infty}^{k'+i\infty}\frac{\chi(s)}{s-w}\,ds = 0 \quad (u > k').$$

Moving the line of integration from k' to $k > u$,

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{\chi(s)}{s-w}\,ds = \chi(w) \quad (u < k). \tag{11.16.11}$$

From (11.16.10), (11.16.11)

$$\chi(w) = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\frac{\mathfrak{G}(1-s)}{s-w}\,ds \quad (u < k), \tag{11.16.12}$$

and this, with (11.16.7) and Mellin's inversion formula for $f(x)$, gives the solution.

If $g(\rho) = v_0 = \text{const.}$, then

$$\mathfrak{G}(1-s) = v_0 \int\limits_0^1 \rho^{-s}\, ds = \frac{v_0}{1-s},$$

$$\chi(w) = \frac{v_0}{2\pi i} \int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{(1-s)\Gamma(\frac{1}{2}s)} \frac{ds}{s-w} = \frac{v_0}{\sqrt{\pi}(1-w)}$$

(from the pole at $s = 1$). Hence

$$\mathfrak{F}(s) = \frac{v_0}{\sqrt{\pi}(1-s)} \frac{2^s\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)} = \frac{v_0\, 2^{s-1}\Gamma(\frac{1}{2}s)}{\sqrt{\pi}\,\Gamma(\frac{3}{2}-\frac{1}{2}s)},$$

and, by (7.9.6),
$$f(u) = \frac{2v_0}{\pi} \frac{\sin u}{u}. \tag{11.16.13}$$

Hence

$$v = \frac{2v_0}{\pi} \int\limits_0^\infty e^{-zu} J_0(\rho u) \frac{\sin u}{u}\, du$$

$$= \frac{2v_0}{\pi} \arcsin\left\{\frac{2}{\sqrt{\{(\rho-1)^2+z^2\}}+\sqrt{\{(\rho+1)^2+z^2\}}}\right\},$$

the solution obtained by Weber.

The pair of equations†

$$\int\limits_0^\infty y^\alpha f(y) J_\nu(xy)\, dy = g(x) \quad (0 < x < 1), \tag{11.16.14}$$

$$\int\limits_0^\infty f(y) J_\nu(xy)\, dy = 0 \qquad (x > 1) \tag{11.16.15}$$

can be solved in a similar way. They are equivalent to

$$\frac{1}{2\pi i} \int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)} \chi(s) x^{s-1-\alpha}\, ds = g(x) \quad (0 < x < 1), \tag{11.16.16}$$

$$\frac{1}{2\pi i} \int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)} \chi(s) x^{s-1}\, ds = 0 \qquad (x > 1), \tag{11.16.17}$$

where
$$\mathfrak{F}(s) = 2^{s-\alpha} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)} \chi(s).$$

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† See King (1).

Multiplying (11.16.16) by $x^{\alpha-w}$, where $\sigma-u > 0$, and integrating over $(0, 1)$, we obtain

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\frac{\chi(s)}{s-w}\,ds = \int\limits_{0}^{1} g(x)x^{\alpha-w}\,dx = \mathfrak{G}(\alpha-w+1).$$

Moving the line of integration to $\sigma = k' < u$, we obtain

$$\frac{1}{2\pi i}\int\limits_{k'-i\infty}^{k'+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\frac{\chi(s)}{s-w}\,ds$$
$$= -\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}w)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}\chi(w)+\mathfrak{G}(\alpha-w+1).$$

Hence the right-hand side is regular for $u > 0$, and we deduce as before that

$$\frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \left\{\chi(s)-\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}\mathfrak{G}(\alpha-s+1)\right\}\frac{ds}{s-w} = 0 \quad (u < k).$$

From (11.16.17) we deduce (11.16.11) as before. Hence

$$\chi(w) = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}\,\frac{\mathfrak{G}(\alpha+1-s)}{s-w}\,ds$$
$$= \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}\,ds\int\limits_{0}^{1} g(\lambda)\lambda^{\alpha-s}\,d\lambda\int\limits_{0}^{1}\mu^{s-w-1}\,d\mu$$
$$= \frac{1}{2\pi i}\int\limits_{0}^{1} g(\lambda)\lambda^{\alpha}\,d\lambda\int\limits_{0}^{1}\mu^{-w-1}\,d\mu\int\limits_{k-i\infty}^{k+i\infty}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}\left(\frac{\lambda}{\mu}\right)^{-s}ds$$
$$= \frac{2}{\Gamma(\frac{1}{2}\alpha)}\int\limits_{0}^{1} g(\lambda)\lambda^{\alpha}\,d\lambda\int\limits_{\lambda}^{1}\mu^{-w-1}\left(\frac{\lambda}{\mu}\right)^{\nu-\alpha+1}\left(1-\frac{\lambda^2}{\mu^2}\right)^{\frac{1}{2}\alpha-1}d\mu$$
$$= \frac{2}{\Gamma(\frac{1}{2}\alpha)}\int\limits_{0}^{1}\mu^{-w-1}\,d\mu\int\limits_{0}^{\mu} g(\lambda)\lambda^{\alpha}\left(\frac{\lambda}{\mu}\right)^{\nu-\alpha+1}\left(1-\frac{\lambda^2}{\mu^2}\right)^{\frac{1}{2}\alpha-1}d\lambda$$
$$= \frac{2}{\Gamma(\frac{1}{2}\alpha)}\int\limits_{0}^{1}\mu^{\alpha-w}\,d\mu\int\limits_{0}^{1} g(\rho\mu)\rho^{\nu+1}(1-\rho^2)^{\frac{1}{2}\alpha-1}\,d\rho.$$

Hence

$$f(x) = \frac{1}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \frac{2^{s-\alpha}\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\chi(s)x^{-s}\,ds$$

$$= \frac{1}{\pi i\Gamma(\frac{1}{2}\alpha)}\int\limits_0^1 \mu^{\alpha}\,d\mu \int\limits_0^1 g(\rho\mu)\rho^{\nu+1}(1-\rho^2)^{\frac{1}{2}\alpha-1}\,d\rho \,\times$$

$$\times \int\limits_{k-i\infty}^{k+i\infty} 2^{s-\alpha}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}(\mu x)^{-s}\,ds$$

$$= \frac{(2x)^{1-\frac{1}{2}\alpha}}{\Gamma(\frac{1}{2}\alpha)}\int\limits_0^1 \mu^{1+\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha}(\mu x)\,d\mu \int\limits_0^1 g(\rho\mu)\rho^{\nu+1}(1-\rho^2)^{\frac{1}{2}\alpha-1}\,d\rho.$$

For this form of the solution to hold we must suppose that $\alpha > 0$; the previous equations correspond to $\nu = 0$, $\alpha = -1$.

As an example, let $\alpha = 1$, $\nu = 0$, $g(x) = 1$; the solution is

$$f(x) = \frac{2}{\pi}\left(\frac{\sin x}{x^2}-\frac{\cos x}{x}\right).$$

11.17. The method of Hopf and Wiener.[†]

A method of Hopf and Wiener for solving the homogeneous equation

$$f(x) = \int\limits_0^\infty k(x-y)f(y)\,dy \quad (0 < x < \infty) \tag{11.17.1}$$

will now be given. It depends on the following lemma.

LEMMA. *Let* $\phi(w)$ *be an analytic function, regular in the strip* $-1 < v < 1$, *and let*

$$\int\limits_{-\infty}^{\infty} |\phi(u+iv)|^2\,du < K = K(\alpha)$$

in any interior strip $-1 < -\alpha \leqslant v \leqslant \alpha < 1$ (*so that, in particular, by the lemma of* § 5.4, $\phi(u+iv) \to 0$ *as* $u \to \pm\infty$ *uniformly in any interior strip*).

In any interior strip $-1 < -\beta \leqslant v \leqslant \beta < 1$, $1-\phi(w)$ *has only a finite number of zeros. If they are* $w_1,\ldots, w_n$, *we can write*

$$1-\phi(w) = \frac{\phi_1(w)}{\phi_2(w)}(w-w_1)\ldots(w-w_n), \tag{11.17.2}$$

[†] Wiener and Hopf (1); Hopf, *Radiative Equilibrium*, Chap. IV; Paley and Wiener, *Fourier Transforms*, Chap. IV.

where $\phi_1(w)$ is regular and free from zeros in $v \leqslant \beta$, $\phi_2(w)$ is regular and free from zeros in $v \geqslant -\beta$, and, in their respective half-planes of regularity,

$$|\phi_1(w)| > K|w|^{-\frac{1}{2}n-k}, \qquad |\phi_2(w)| > K|w|^{\frac{1}{2}n-k}, \tag{11.17.3}$$

where k is a positive integer depending on ϕ.

Let
$$\psi(w) = \{1-\phi(w)\}\frac{(w^2+1)^{\frac{1}{2}n}}{(w-w_1)...(w-w_n)}\left(\frac{w-i}{w+i}\right)^k,$$
where $(w^2+1)^{\frac{1}{2}n}$ is that single-valued branch in the strip $-\beta \leqslant v \leqslant \beta$ which behaves like w^n for large w, and where k is an integer still to be determined. Then $\psi(w) \to 1$ as $u \to \pm\infty$. Hence we can choose k so that the variation of $\log\psi(w)$ along the whole strip is 0. Having fixed k, let $\log\psi(w)$ denote the branch which tends to 0 as $u \to \pm\infty$. Since
$$\psi(w) = \{1-\phi(w)\}\left\{1+O\left(\frac{1}{|w|}\right)\right\},$$
$|\log\psi(w)|$ belongs to L^2 uniformly in the strip. Hence
$$\log\psi(w) = \frac{1}{2\pi i}\int\limits_{-i\gamma-\infty}^{-i\gamma+\infty}\frac{\log\psi(z)}{z-w}\,dz - \frac{1}{2\pi i}\int\limits_{i\gamma-\infty}^{i\gamma+\infty}\frac{\log\psi(z)}{z-w}\,dz$$
$$= \chi_1(w)-\chi_2(w) \qquad (-\gamma < v < \gamma),$$
where $0 < \beta < \gamma < 1$, but $\gamma-\beta$ is so small that no zeros of $\psi(w)$ lie in $\beta < v \leqslant \gamma$. Now $\chi_1(w)$ is regular for $v > -\gamma$, and regular and bounded for $v \geqslant -\beta$; and similarly χ_2 for $v \leqslant \beta$. Since
$$1-\phi(w) = \frac{e^{\chi_1(w)}}{e^{\chi_2(w)}}\frac{(w-i)^{-\frac{1}{2}n-k}}{(w+i)^{\frac{1}{2}n-k}}(w-w_1)...(w-w_n),$$
the result now follows.

Suppose now that $f(x)$ is a function which satisfies (11.17.1) for *all* real x, and is $O(e^{cx})$ as $x \to \infty$, where $0 < c < 1$; and let $k(x) = O(e^{-|x|})$, or more generally let $e^{\lambda|x|}k(x)$ belong to $L^2(-\infty,\infty)$ for all $\lambda < 1$. Then as $x \to -\infty$
$$f(x) = O\left\{\int\limits_0^\infty e^{\lambda(x-y)}|h(x-y)|e^{cy}\,dy\right\}$$
for any $\lambda < 1$, where $h(y)$ is $L^2(-\infty,\infty)$; and by choosing $\lambda > c$, and applying Schwarz's inequality, we have
$$f(x) = O(e^{\lambda x})$$
for any $\lambda < 1$. Thus $F_+(w)$ is regular for $v > c$, $F_-(w)$ is regular for $v < 1$, and $K(w)$ is regular for $-1 < v < 1$.

Now

$$\int_0^\infty k(x-y)f(y)\,dy = \int_{-\infty}^\infty k(x-y)f_1(y)\,dy$$

(where $f_1(y) = f(y)$ $(y > 0)$, 0 $(y < 0)$)

$$= \int_{ia-\infty}^{ia+\infty} F_+(w)K(w)e^{-ixw}\,dw \quad (c < a < 1),$$

by Theorem 64. The equation (11.17.1) therefore gives

$$\int_{ia-\infty}^{ia+\infty} F_+(w)\{1-\sqrt{(2\pi)}K(w)\}e^{-ixw}\,dw + \int_{ia-\infty}^{ia+\infty} F_-(w)e^{-ixw}\,dw = 0.$$

Here each integrand is regular throughout the strip $c < v < 1$, and so in this strip we must have

$$F_+(w)\{1-\sqrt{(2\pi)}K(w)\}+F_-(w) = 0,$$

which implies that in fact each term is regular for $v < 1$. If $w_1,\ldots,w_n$ are the zeros of $1-\sqrt{(2\pi)}K(w)$ in $-\beta \leqslant v \leqslant \beta$, by the lemma

$$F_+(w)\frac{\phi_1(w)}{\phi_2(w)}(w-w_1)\ldots(w-w_n)+F_-(w) = 0,$$

where ϕ_1 and ϕ_2 have the properties stated in the lemma. We can write this

$$\frac{F_+(w)}{\phi_2(w)}(w-w_1)\ldots(w-w_n) = -\frac{F_-(w)}{\phi_1(w)},$$

and here the left-hand side is regular for $v \geqslant -\beta$, the right-hand side for $v \leqslant \beta$. Hence each side is an integral function, and by (11.17.3) this must be a polynomial of degree not exceeding $\frac{1}{2}n+k$. Hence

$$F_+(w) = \frac{\phi_2(w)P(w)}{(w-w_1)\ldots(w-w_n)},$$

where $P(w)$ is a polynomial. Hence

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} \frac{\phi_2(w)P(w)}{(w-w_1)\ldots(w-w_n)}e^{-ixw}\,dw$$

satisfies the original equation (and vanishes for $x < 0$).

As a simple example, let

$$k(x) = \lambda e^{-|x|} \quad (0 < \lambda < \tfrac{1}{2}),$$

$$K(w) = \frac{\lambda}{\sqrt{(2\pi)}}\int_{-\infty}^\infty e^{-|x|+ixw}\,dx = \frac{2\lambda}{\sqrt{(2\pi)}}\frac{1}{1+w^2},$$

$$1-\sqrt{(2\pi)}K(w) = 1-\frac{2\lambda}{1+w^2} = \frac{w^2-(2\lambda-1)}{w^2+1}.$$

The roots are $w = \pm\sqrt{(2\lambda-1)} = w_1, w_2$, and

$$1-\sqrt{(2\pi)}K(w) = \frac{(w-w_1)(w-w_2)}{(w-i)(w+i)},$$

$$\phi_1(w) = \frac{1}{w-i}, \qquad \phi_2(w) = w+i,$$

$$F_+(w) = \frac{(w+i)P(w)}{w^2-2\lambda+1}, \qquad P(w) = \text{const.},$$

$$f(x) = \frac{C}{\sqrt{(2\pi)}} \int\limits_{ia-\infty}^{ia+\infty} \frac{w+i}{w^2-2\lambda+1} e^{-ixw}\, dw$$

$$= C'\left\{\frac{1+\sqrt{(1-2\lambda)}}{2\sqrt{(1-2\lambda)}} e^{x\sqrt{(1-2\lambda)}} - \frac{1-\sqrt{(1-2\lambda)}}{2\sqrt{(1-2\lambda)}} e^{-x\sqrt{(1-2\lambda)}}\right\}.$$

11.18. An equation of A. C. Dixon. A similar problem is presented by the equation†

$$f(x) = g(x)+\lambda \int\limits_0^1 \frac{f(t)}{x+t}\, dt. \tag{11.18.1}$$

This is satisfied formally by

$$f(x) = g(x)+\lambda \int\limits_0^1 g(t)\chi(x,t)\, dt \tag{11.18.2}$$

if $\chi(x,t)$ satisfies the integral equation

$$\chi(x,t) = \frac{1}{x+t}+\lambda \int\limits_0^1 \frac{\chi(y,t)}{x+y}\, dy. \tag{11.18.3}$$

Putting $x = e^{-\xi}$, $y = e^{-\eta}$, $t = e^{-\beta}$, this is

$$e^{-\frac{1}{2}\xi}\chi(e^{-\xi},e^{-\beta}) = \frac{e^{-\frac{1}{2}\xi}}{e^{-\xi}+e^{-\beta}}+\lambda \int\limits_0^\infty \frac{e^{-\frac{1}{2}\eta}\chi(e^{-\eta},e^{-\beta})}{2\cosh\frac{1}{2}(\xi-\eta)}\, d\eta,$$

or, writing $e^{-\frac{1}{2}\xi}\chi(e^{-\xi},e^{-\beta}) = \phi(\xi)$,

$$\phi(\xi) = \frac{e^{-\frac{1}{2}\xi}}{e^{-\xi}+e^{-\beta}}+\lambda \int\limits_0^\infty \frac{\phi(\eta)}{2\cosh\frac{1}{2}(\xi-\eta)}\, d\eta. \tag{11.18.4}$$

Suppose that $\phi(\xi) = O(e^{c\xi})$ as $\xi \to \infty$, where $0 < c < \frac{1}{2}$. Then, as in § 11.17, $\phi(\xi) = O(e^{\frac{1}{2}\xi})$ as $\xi \to -\infty$. Let $c < a < \frac{1}{2}$. Then $\Phi_+(w)$ is regular for $v > c$, $\Phi_-(w)$ is regular for $v < \frac{1}{2}$, and so (11.18.4) is equivalent to

$$\int\limits_{ia-\infty}^{ia+\infty} \Phi_+(w)e^{-i\xi w}\, dw + \int\limits_{ia-\infty}^{ia+\infty} \Phi_-(w)e^{-i\xi w}\, dw$$

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† A. C. Dixon (2).

$$= \sqrt{\left(\frac{\pi}{2}\right)} \int\limits_{ib-\infty}^{ib+\infty} \frac{e^{\frac{1}{2}\beta+iw\beta}}{\cosh \pi w} e^{-i\xi w}\, dw + \pi\lambda \int\limits_{ia-\infty}^{ia+\infty} \frac{\Phi_+(w)}{\cosh \pi w} e^{-i\xi w}\, dw,$$

where $-\frac{1}{2} < b < \frac{1}{2}$. We can take $b = a$. It follows that

$$\Phi_+(w)\left(1-\frac{\pi\lambda}{\cosh \pi w}\right)-\sqrt{\left(\frac{\pi}{2}\right)}\frac{e^{\frac{1}{2}\beta+i\beta w}}{\cosh \pi w} \tag{11.18.5}$$

and $-\Phi_-(w)$ are regular and equal in the strip $-a < v < a$. Hence (11.18.5) is regular for $v < \frac{1}{2}$. Hence $\Phi_+(w)$ is regular for $v < \frac{1}{2}$, except possibly for simple poles at the zeros of $\cosh \pi w - \pi\lambda$. Suppose for example that $\pi\lambda = \sin \alpha\pi$, where $0 < \alpha < \frac{1}{2}$. Then the zeros are at

$$(\tfrac{1}{2}-\alpha)i,\ (-\tfrac{3}{2}-\alpha)i,\ldots,\ (-\tfrac{1}{2}+\alpha)i,\ (-\tfrac{5}{2}+\alpha)i,\ldots.$$

Hence
$$\Psi(w) = \frac{\Phi_+(w)}{\Gamma(-\frac{1}{4}+\frac{1}{2}\alpha-\frac{1}{2}iw)\Gamma(\frac{1}{4}-\frac{1}{2}\alpha-\frac{1}{2}iw)}$$

is regular for $v < \frac{1}{2}$.

To cancel the poles of $\operatorname{sech} \pi w$ in (11.18.5) we must also have

$$\Phi_+(w) = -\frac{1}{\lambda\sqrt{(2\pi)}} e^{\frac{1}{2}\beta+i\beta w}$$

at $w = -\frac{1}{2}i,\ldots, -(n+\frac{1}{2})i,\ldots$. Hence

$$\Psi\{-(n+\tfrac{1}{2})i\} = -\frac{1}{\lambda\sqrt{(2\pi)}}\frac{e^{(n+1)\beta}}{\Gamma(-\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}n)\Gamma(-\frac{1}{2}\alpha-\frac{1}{2}n)} = a_n,$$

say. The most obvious function with these properties is

$$\Psi(w) = \frac{i}{\Gamma(\frac{1}{2}-iw)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{a_n}{w+(n+\frac{1}{2})i},$$

and it is easily verified that this does in fact give a solution.

The difference between two solutions of (11.18.4) satisfies

$$\phi(\xi) = \lambda \int\limits_0^{\infty} \frac{\phi(\eta)}{2\cosh\frac{1}{2}(\xi-\eta)}\, d\eta, \tag{11.18.6}$$

which is of the form (11.17.1). Here

$$1-\sqrt{(2\pi)}K(w) = 1-\frac{\pi\lambda}{\cosh \pi w}$$
$$= \frac{2\pi\Gamma(\frac{1}{2}-iw)\Gamma(\frac{1}{2}+iw)}{\Gamma(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{2}iw)\Gamma(\frac{1}{4}-\frac{1}{2}\alpha-\frac{1}{2}iw)\Gamma(\frac{3}{4}+\frac{1}{2}\alpha+\frac{1}{2}iw)\Gamma(\frac{3}{4}+\frac{1}{2}\alpha-\frac{1}{2}iw)}$$
$$= \frac{\phi_1(w)}{\phi_2(w)}(\tfrac{1}{4}-\tfrac{1}{2}\alpha+\tfrac{1}{2}iw)(\tfrac{1}{4}-\tfrac{1}{2}\alpha-\tfrac{1}{2}iw),$$

where

$$\phi_1(w) = \frac{2\pi\Gamma(\frac{1}{2}+iw)}{\Gamma(\frac{5}{4}-\frac{1}{2}\alpha+\frac{1}{2}iw)\Gamma(\frac{3}{4}+\frac{1}{2}\alpha+\frac{1}{2}iw)},$$

$$\phi_2(w) = \frac{\Gamma(\frac{5}{4}-\frac{1}{2}\alpha-\frac{1}{2}iw)\Gamma(\frac{3}{4}+\frac{1}{2}\alpha-\frac{1}{2}iw)}{\Gamma(\frac{1}{2}-iw)}$$

have the properties stated in § 11.17.

11.19. A problem of radiative equilibrium.† Consider a medium stratified in planes perpendicular to the axis of x, extending indefinitely on the positive side of its boundary $x = 0$.

Let I (a function of x and θ) be the intensity of radiation of all wave-lengths, at any point, in a direction making an angle θ with the negative direction of the axis of x. Let ρ be the density at any point, and k the coefficient of mass-absorption, supposed independent of the wave-length. Let B (a function of x) be the intensity of black-body radiation corresponding to the temperature of the matter at x.

The rate of absorption of energy per unit volume from the radiation in a solid angle ω is

$$k\rho \iint I\,d\omega,$$

while the rate of emission is

$$k\rho \iint B\,d\omega = k\rho B\omega.$$

Consider a narrow circular cylinder, area of cross-section a, the centres of whose ends are at x and x', and whose axis makes an angle θ with the negative x-axis. The energy radiated from the x'-end through a distant area in the line of the axis of the cylinder, at which all points of the cylinder subtend approximately the same small solid angle ω, is $I(x',\theta)a\omega$; this is made up of $I(x,\theta)a\omega$ from the x-end, together with

$$\int k\rho(B-I)\omega\,dv$$

from the interior of the cylinder, v being its element of volume.

In the limit as $a \to 0$, $\omega \to 0$ we obtain

$$I(x',\theta) = I(x,\theta) - \int\limits_{x}^{x'} k\rho(B-I)\sec\theta\,d\xi,$$

and, making $x' \to x$,

$$\frac{\partial I}{\partial x} = k\rho\sec\theta(I-B). \qquad (11.19.1)$$

For radiative equilibrium, the rate of absorption of energy per unit volume from all directions is equal to the rate of emission in all directions; this gives

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† E. A. Milne (1), Hopf (3); Hopf, *Radiative Equilibrium*.

$$4\pi k\rho B = k\rho \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} I \sin\theta \, d\theta$$

$$= 2\pi k\rho \int\limits_0^{\pi} I \sin\theta \, d\theta,$$

i.e.
$$2B = \int\limits_0^{\pi} I \sin\theta \, d\theta. \qquad \text{(11 19.2)}$$

Putting
$$\tau = \int\limits_0^{x} k\rho \, dx,$$

(11.19.1) becomes
$$\frac{\partial I}{\partial \tau} = \sec\theta(I - B). \qquad (11.19.3)$$

Hence
$$I = e^{\tau\sec\theta}\left\{K - \int\limits^{\tau} B(t)\sec\theta \, e^{-t\sec\theta} \, dt\right\}.$$

The boundary condition is that the incident radiation is zero, i.e. that $I = 0$ for $x = 0$, $\frac{1}{2}\pi < \theta \leqslant \pi$. Hence

$$I = -e^{-\tau\sec\theta} \int\limits_0^{\tau} B(t)\sec\theta \, e^{-t\sec\theta} \, dt \quad (\tfrac{1}{2}\pi < \theta \leqslant \pi). \qquad (11.19.4)$$

For $0 < \theta \leqslant \frac{1}{2}\pi$ we choose K so that I is not exponentially large as $\tau \to \infty$, i.e. we obtain

$$I = e^{\tau\sec\theta} \int\limits_{\tau}^{\infty} B(t)\sec\theta \, e^{-t\sec\theta} \, dt \quad (0 < \theta \leqslant \tfrac{1}{2}\pi).$$

Inserting these results in (11.19.2), we obtain

$$B(\tau) = \tfrac{1}{2} \int\limits_0^{\frac{1}{2}\pi} e^{\tau\sec\theta} \sin\theta \, d\theta \int\limits_{\tau}^{\infty} B(t)\sec\theta \, e^{-t\sec\theta} \, dt -$$

$$-\tfrac{1}{2} \int\limits_{\frac{1}{2}\pi}^{\pi} e^{\tau\sec\theta} \sin\theta \, d\theta \int\limits_0^{\tau} B(t)\sec\theta \, e^{-t\sec\theta} \, dt$$

$$= \tfrac{1}{2} \int\limits_{\tau}^{\infty} B(t) \, dt \int\limits_0^{\frac{1}{2}\pi} e^{(\tau - t)\sec\theta} \tan\theta \, d\theta -$$

$$-\tfrac{1}{2} \int\limits_0^{\tau} B(t) \, dt \int\limits_{\frac{1}{2}\pi}^{\pi} e^{(\tau - t)\sec\theta} \tan\theta \, d\theta$$

$$= \int_0^\infty B(t)k(\tau-t)\,dt, \tag{11.19.5}$$

where
$$k(\xi) = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} e^{-|\xi|\sec\theta}\tan\theta\,d\theta = \tfrac{1}{2}\int_{|\xi|}^\infty \frac{e^{-\lambda}}{\lambda}\,d\lambda.$$

We can now appeal to the theory of § 11.17. We have, if $v < 1$,

$$K(w) = \frac{1}{2\sqrt{(2\pi)}}\int_{-\infty}^\infty e^{ixw}\,dx\int_{|x|}^\infty \frac{e^{-\lambda}}{\lambda}\,d\lambda = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \cos xw\,dw\int_x^\infty \frac{e^{-\lambda}}{\lambda}\,d\lambda$$

$$= \frac{1}{\sqrt{(2\pi)}w}\int_0^\infty \sin xw\frac{e^{-x}}{x}\,dx = \frac{1}{\sqrt{(2\pi)}}\frac{\arctan w}{w},$$

so that
$$1-\sqrt{(2\pi)}K(w) = 1-\frac{\arctan w}{w}.$$

This has a double zero at the origin, and no other zeros in the strip $-1 < v < 1$. Hence, with the notation of § 11.17, we put

$$\psi(w) = \left(1-\frac{\arctan w}{w}\right)\frac{w^2+1}{w^2},$$

no additional factor being needed. Hence

$$\chi_2(w) = \frac{1}{2\pi i}\int_{i\gamma-\infty}^{i\gamma+\infty} \log\left\{\left(1-\frac{\arctan z}{z}\right)\frac{z^2+1}{z^2}\right\}\frac{dz}{z-w} \quad (v < \gamma).$$

Also $P(w) = \alpha+\beta w$, where α and β are constants, and the solution is

$$B(\tau) = \frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} \frac{\alpha+\beta w}{w^2}e^{-i\tau w+\chi_2(w)}\,dw.$$

11.20. The limiting form of Milne's equation.[†] Writing $\int_0^x B(\tau)\,d\tau = f(x)$, (11.19.5) may also be written

$$f'(x) = \tfrac{1}{2}\int_0^\infty f'(t)\,dt\int_{|x-t|}^\infty \frac{e^{-y}}{y}\,dy$$

$$= \tfrac{1}{2}\int_0^\infty \frac{e^{-y}}{y}\,dy\int_{\max(0,x-y)}^{x+y} f'(t)\,dt$$

[†] E. A. Milne (1), Hardy and Titchmarsh (1), (2). See also Hopf (2).

$$= \tfrac{1}{2}\int\limits_{0}^{x} \frac{e^{-y}}{y}\{f(x+y)-f(x-y)\}\,dy+\tfrac{1}{2}\int\limits_{x}^{\infty}\frac{e^{-y}}{y}f(x+y)\,dy. \tag{11.20.1}$$

For large values of x this approximates to the form

$$f'(x) = \tfrac{1}{2}\int\limits_{0}^{\infty}\frac{e^{-y}}{y}\{f(x+y)-f(x-y)\}\,dy. \tag{11.20.2}$$

Theorem 155. *If $f(x) = O(e^{c|x|})$, where $0 < c < 1$, and both sides of* (11.20.2) *are finite and equal for every x, then $f(x)$ is a quadratic.*

The formal argument is similar to that of § 11.2. We have

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)e^{-ixw}\,dw+\frac{1}{\sqrt{(2\pi)}}\int\limits_{ib-\infty}^{ib+\infty} F_{-}(w)e^{-ixw}\,dw, \tag{11.20.3}$$

where $1 > a > c$, $-1 < b < -c$. Hence

$$f(x+y)-f(x-y) = -\frac{2i}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)\sin yw\, e^{-ixw}\,dw-\dots,$$

$$\int\limits_{0}^{\infty}\frac{e^{-y}}{y}\{f(x+y)-f(x-y)\}\,dy$$

$$= -\frac{2i}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)e^{-ixw}\,dw\int\limits_{0}^{\infty}\frac{e^{-y}}{y}\sin yw\,dy-\dots$$

$$= -\frac{2i}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)\arctan w\, e^{-ixw}\,dw-\dots, \tag{11.20.4}$$

the dots indicating in each case the corresponding term involving $F_{-}(w)$. Also

$$f'(x) = -\frac{i}{\sqrt{(2\pi)}}\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)w\, e^{-ixw}\,dw-\dots. \tag{11.20.5}$$

Hence (11.20.2) gives

$$\int\limits_{ia-\infty}^{ia+\infty} F_{+}(w)(w-\arctan w)e^{-ixw}\,dw+\dots = 0.$$

Hence, by Theorem 141, p. 255, $F_{+}(w)$ and $F_{-}(w)$ are regular for $b \leqslant v \leqslant a$, except possibly for a triple pole at the origin corresponding to the triple zero of $w-\arctan w$; and $F_{+}(w) = -F_{-}(w)$. Evaluating (11.20.3) by the calculus of residues, it follows that $f(x)$ is a quadratic.

To justify the process we shall first prove that $e^{-c'|x|}f'(x)$ *belongs to* $L^2(-\infty,\infty)$ *if* $c' > c$. For (11.20.2) gives

$$f'(x) = \tfrac{1}{2}\int_0^1 \frac{f(x+y)-f(x-y)}{y}\,dy +$$

$$+\tfrac{1}{2}\left[\int_0^1 \frac{e^{-y}-1}{y}\{f(x+y)-f(x-y)\}\,dy + \int_1^\infty \frac{e^{-y}}{y}\{f(x+y)-f(x-y)\}\,dy\right]$$

$$= \tfrac{1}{2}\phi(x)+\tfrac{1}{2}\psi(x),$$

say. If $|f(x)| \leqslant Ke^{c|x|}$,

$$|\psi(x)| \leqslant K\int_0^1 (e^{c|x+y|}+e^{c|x-y|})\,dy + \int_1^\infty e^{-y}(e^{c|x+y|}+e^{c|x-y|})\,dy < Ke^{c|x|}.$$

We may write
$$\phi(x) = \int_{x-1}^{x+1} \frac{f(t)}{t-x}\,dt,$$

where the integral is a principal value at $t = x$. We now appeal to the theory of conjugate functions. Let

$$\phi_1(x) = \int_{\xi-2}^{\xi+2} \frac{f(t)}{t-x}\,dt.$$

Then
$$\int_{\xi-1}^{\xi+1} |\phi_1(x)|^2\,dx \leqslant \int_{-\infty}^{\infty} |\phi_1(x)|^2\,dx = \pi^2\int_{\xi-2}^{\xi+2} |f(t)|^2\,dt$$

by (5.3.3). Also, for $\xi-1 < x < \xi+1$,

$$|\phi_1(x)-\phi(x)| \leqslant \left|\int_{x+1}^{\xi+2} \frac{f(t)}{t-x}\,dt\right| + \left|\int_{\xi-2}^{x-1} \frac{f(t)}{t-x}\,dt\right|$$

$$\leqslant \left(\int_{x+1}^{\xi+2} \frac{dt}{(t-x)^2}\int_{x+1}^{\xi+2} |f(t)|^2\,dt\right)^{\frac{1}{2}} + \left(\int_{\xi-2}^{x-1} \frac{dt}{(t-x)^2}\int_{\xi-2}^{x-1} |f(t)|^2\,dt\right)^{\frac{1}{2}}$$

$$< A\left(\int_{\xi-2}^{\xi+2} |f(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Altogether it follows that

$$\int_{\xi-1}^{\xi+1} |\phi(x)|^2\,dx < A \int_{\xi-2}^{\xi+2} |f(x)|^2\,dx < K\,e^{2c|\xi|}.$$

Hence

$$\int_{\xi-1}^{\xi+1} |f'(x)|^2\,dx < K\,e^{2c|\xi|},$$

$$\int_{\xi-1}^{\xi+1} e^{-2c'|x|}|f'(x)|^2\,dx < K\,e^{2(c-c')|\xi|},$$

and the result stated follows.

It now follows that the integrals (11.20.5) exist in the mean-square sense, $wF_+(w)$ being $L^2(ia-\infty, ia+\infty)$ if $a > c$. Also the inversion of (11.20.4) is justified by absolute convergence; for $\sin yw$ is $O(e^{av})$ for all y, and $O(|yw|)$ for small $|yw|$, and so is

$$O(|yw|^{\frac{1}{2}}e^{av})$$

for all y and w; and

$$\int_{ia-\infty}^{ia+\infty} |F_+(w)w^{\frac{1}{2}}\,dw| \int_0^\infty y^{-\frac{1}{2}}e^{(a-1)y}\,dy$$

is convergent. This completes the proof.

It has been proved by a more complicated method[†] that the result holds under less restrictive assumptions.

11.21. Bateman's equation.[‡] Suppose that a function $f(x)$ is represented by Fourier's single-integral formula (1.1.7), not merely in the limit, but for some value of λ, $\lambda = a$ say, exactly. Then

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{\sin a(x-y)}{x-y}\,dy \tag{11.21.1}$$

for a given a and all x.

This is an integral equation of the form (11.2.1), but the conditions if § 11.2 are not satisfied, and the solution takes quite a different form.

Suppose that $f(x)$ belongs to $L^2(-\infty,\infty)$. Let

$$g(x) = \sin ax/x, \qquad G(x) = \sqrt{(\tfrac{1}{2}\pi)}\ (|x| < a), \quad 0\ (|x| > a).$$

[†] Hardy and Titchmarsh (9).

[‡] Bateman (1), Hardy (2), Hardy and Titchmarsh (1), (2).

Then (2.1.8) gives

$$\int_{-\infty}^{\infty} f(y)\frac{\sin a(x-y)}{x-y}\,dy = \sqrt{\left(\frac{\pi}{2}\right)}\int_{-a}^{a} F(t)e^{-ixt}\,dt. \tag{11.21.2}$$

Hence

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a} F(t)e^{-ixt}\,dt, \tag{11.21.3}$$

i.e. $f(x)$ is a finite trigonometrical integral. Conversely, if $f(x)$ is of the form (11.21.3), where F is L^2, (11.21.1) follows from (11.21.2). Hence

Theorem 156. *A necessary and sufficient condition that a function $f(x)$ of L^2 should be a solution of* (11.21.1) *is that it should be of the form* (11.21.3), *where F is $L^2(-a,a)$.*

There are, however, simple solutions of (11.21.1) not belonging to L^2; for example, $\cos bx$ and $\sin bx$ are solutions if $-a < b < a$, though not if $|b| > |a|$. The next theorem includes these solutions.

Theorem 157. *Let $f(x)/(1+|x|)$ belong to $L^2(-\infty,\infty)$, and let*

$$\int^{\to\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, ax\frac{f(x)}{x}\,dx, \qquad \int_{\to-\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, ax\frac{f(x)}{x}\,dx$$

exist. Then, if $f(x)$ satisfies (11.21.1), *it is of the form*

$$f(x) = f(0)+x\int_{-a}^{a} \chi(u)e^{-ixu}\,du,$$

where $\chi(u)$ belongs to $L^2(-a,a)$.

It is easily verified that

$$\int_{-a}^{a} (e^{ivu}-e^{iva\operatorname{sgn} u})e^{-ixu}\,du = \frac{2y}{x}\left\{\frac{\sin a(x-y)}{x-y}-\frac{\sin ay}{y}\right\}.$$

Hence

$$f(x)-f(0) = \frac{1}{\pi}\int_{\to-\infty}^{\to\infty} \left\{\frac{\sin a(x-y)}{x-y}-\frac{\sin ay}{y}\right\}f(y)\,dy$$

$$= \frac{x}{2\pi}\int_{\to-\infty}^{\to\infty} \frac{f(y)}{y}\,dy\int_{-a}^{a} (e^{ivu}-e^{iva\operatorname{sgn} u})e^{-ixu}\,du$$

$$= \frac{x}{2\pi}\int_{-a}^{a} e^{-ixu}\,du \int_{\to-\infty}^{\to\infty} \frac{f(y)}{y}(e^{iyu}-e^{iya\,\mathrm{sgn}\,u})\,dy$$

if we can invert the order of integration. This is obviously permissible for the part with $|y| < 1$, and for the part involving $e^{iya\,\mathrm{sgn}\,u}$ and $|y| > 1$. Also $f(y)/y$ is $L^2(-\infty,-1)$ and $L^2(1,\infty)$, and the integrals

$$\int_{-\infty}^{-1} \frac{f(y)}{y} e^{iyu}\,dy, \qquad \int_{1}^{\infty} \frac{f(y)}{y} e^{iyu}\,dy$$

exist in the mean-square sense. The inversion for these is a case of Parseval's formula in L^2 theory. The y-integral represents a function of $L^2(-a,a)$, and this is the result stated.

THEOREM 158. *Let*

$$f(x) = f(0)+x\int_{-a}^{a} \chi(u)e^{-ixu}\,du,$$

where $\chi(u)/(a^2-u^2)$ *is* $L(-a,a)$. *Then*

$$f(x) = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} \frac{\sin ay}{y} f(x-y)\,dy.$$

We may suppose $f(0) = 0$. Then

$$\int_{-\lambda}^{\lambda} \frac{\sin ay}{y} f(x-y)\,dy = \int_{-\lambda}^{\lambda} \frac{\sin ay}{y}(x-y)\,dy \int_{-a}^{a} \chi(u)e^{-i(x-y)u}\,du$$

$$= x\int_{-a}^{a} \chi(u)e^{-ixu}\,du \int_{-\lambda}^{\lambda} \frac{\sin ay}{y} e^{iyu}\,dy - \int_{-a}^{a} \chi(u)e^{-ixu}\,du \int_{-\lambda}^{\lambda} \sin ay\, e^{iyu}\,dy.$$

The first term tends to $\pi x \int_{-a}^{a} e^{-ixu}\chi(u)\,du = \pi f(x)$, by the bounded convergence of the inner integral. The second term is

$$-2i\int_{-a}^{a} e^{-ixu}\chi(u)\,du \int_{0}^{\lambda} \sin ay \sin yu\,dy$$

$$= -i\int_{-a}^{a} e^{-ixu}\chi(u)\left\{\frac{\sin(a-u)\lambda}{a-u} - \frac{\sin(a+u)\lambda}{a+u}\right\}du,$$

which tends to 0 as λ → ∞ with the given conditions. Hence the result.

The function $f(x) = \sin bx$ $(|b| < a)$ is a case of both these theorems; $f(x) = \sin ax$ is not a solution of the equation.

Also, if $0 \leqslant m \leqslant n$, $x^{-m}J_n(x)$ is a solution of the equation with $a = 1$. For

$$\frac{J_n(x)}{x^m} = \frac{x^{n-m}}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1}(1-y^2)^{n-\frac{1}{2}}e^{ixv}\,dy$$

$$= \frac{i^{n-m}}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1}\left(\frac{d}{dy}\right)^{n-m}(1-y^2)^{n-\frac{1}{2}}.e^{ixv}\,dy.$$

11.22. Kapteyn's equation.† A Neumann series (for an odd function) is an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_{2n+1}J_{2n+1}(x). \tag{11.22.1}$$

If $f(x)$ is given, the coefficients a_{2n+1} may be obtained formally as follows. We have (e.g. from (7.10.1))

$$\int_0^{\infty} J_{2m+1}(t)J_{2n+1}(t)\frac{dt}{t} = \begin{matrix} 0 & (m \neq n), \\ 1/(4n+2) & (m = n). \end{matrix} \tag{11.22.2}$$

Hence, multiplying by $J_{2m+1}(t)/t$ and integrating over $(0,\infty)$, we obtain

$$a_{2m+1} = (4m+2)\int_0^{\infty} f(t)\frac{J_{2m+1}(t)}{t}\,dt. \tag{11.22.3}$$

The series formed with these coefficients is

$$\sum_{n=0}^{\infty}(4n+2)J_{2n+1}(x)\int_0^{\infty} f(t)\frac{J_{2n+1}(t)}{t}\,dt$$

$$= \int_0^{\infty}\frac{f(t)}{t}\left\{\sum_{n=0}^{\infty}(4n+2)J_{2n+1}(x)J_{2n+1}(t)\right\}dt$$

(provided that we may integrate term-by-term)

$$= \tfrac{1}{2}\int_0^{\infty} f(t)\,dt\int_0^{x}\left\{\frac{J_1(t-v)}{t-v}+\frac{J_1(t+v)}{t+v}\right\}J_0(x-v)\,dv$$

$$= \tfrac{1}{2}\int_0^{x} J_0(x-v)\,dv\int_0^{\infty} f(t)\left\{\frac{J_1(t-v)}{t-v}+\frac{J_1(t+v)}{t+v}\right\}dt$$

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† See Watson, § 16.4, Hardy and Titchmarsh (1). Also Sears and Titchmarsh (1).

by § 11.12. The inner integral is

$$\int_{-v}^{\infty} f(u+v)\frac{J_1(u)}{u}\,du+\int_{v}^{\infty} f(u-v)\frac{J_1(u)}{u}\,du$$

$$=\int_{0}^{\infty}\{f(u+v)+f(u-v)\}\frac{J_1(u)}{u}\,du+2\int_{0}^{v} f(v-u)\frac{J_1(u)}{u}\,du,$$

and the last term gives

$$\int_{0}^{x} J_0(x-v)\,dv\int_{0}^{v} f(u)\frac{J_1(v-u)}{v-u}\,du=\int_{0}^{x} f(u)\,du\int_{u}^{x} J_0(x-v)\frac{J_1(v-u)}{v-u}\,dv$$

$$=\int_{0}^{x} f(u)J_1(x-u)\,du=f(x)-\int_{0}^{x} f'(u)J_0(x-u)\,du$$

on integrating by parts. The sum of the series is therefore $f(x)$ if

$$\int_{0}^{x} f'(u)J_0(x-u)\,du=\tfrac{1}{2}\int_{0}^{x} J_0(x-u)\,du\int_{0}^{\infty}\{f(\xi+u)+f(\xi-u)\}\frac{J_1(\xi)}{\xi}\,d\xi,$$

and, by Theorem 150, this implies that

$$f'(u)=\tfrac{1}{2}\int_{0}^{\infty}\{f(\xi+u)+f(\xi-u)\}\frac{J_1(\xi)}{\xi}\,d\xi. \tag{11.22.4}$$

This is Kapteyn's integral equation.

11.23. Before proceeding to rigorous analysis, we shall prove the following lemma.

LEMMA. *For $x>0$, $t>0$,*

$$\sum_{n=0}^{\infty}(4n+2)|J_{2n+1}(x)J_{2n+1}(t)|=O\{\min(x^3,x^{\frac12})\min(t^3,t^{-\frac12})\}.$$

We have

$$J_n(x)=O(x^{-\frac12})\qquad(n\leqslant\tfrac{1}{2}x), \tag{11.23.1}$$

$$=O(1)\qquad(\text{all } n \text{ and } x), \tag{11.23.2}$$

and

$$J_n(x)=O\left\{\frac{(\frac{1}{2}x)^n}{n!}\right\}=O\left\{\frac{1}{\sqrt{n}}\left(\frac{xe}{2n}\right)^n\right\}\quad(\text{all } n \text{ and } x), \tag{11.23.3}$$

so that in particular

$$J_n(x)=O(2^{-n})\quad(n\geqslant ex). \tag{11.23.4}$$

For $x > 1$, $2ex \leqslant t$ the above sum is therefore

$$\sum_{2n+1=3}^{ex} O\left(\frac{n}{\sqrt{t}}\right)+\sum_{ex}^{\frac{1}{2}t} O\left(\frac{n}{2^n\sqrt{t}}\right)+\sum_{\frac{1}{2}t}^{\infty} O\left(\frac{n}{2^n}\right)$$

$$= O(x^2t^{-\frac{1}{2}})+O(t^{-\frac{1}{2}})+O(e^{-At}) = O(x^2t^{-\frac{1}{2}}).$$

For $1 < x < t < 2ex$ it is

$$\sum_{2n+1=3}^{ex} O(n)+\sum_{ex}^{\infty} O\left(\frac{n}{2^n}\right) = O(x^2) = O(x^{\frac{5}{2}}t^{-\frac{1}{2}}).$$

For $x \leqslant 1 < t$ it is

$$\sum_{2n+1=3}^{\frac{1}{2}t} O\left\{\frac{x^3}{\sqrt{t}}\sqrt{n}\left(\frac{e}{2^n}\right)^n\right\}+\sum_{\frac{1}{2}t}^{\infty} O\left\{x^3\sqrt{n}\left(\frac{e}{2^n}\right)^n\right\}$$

$$= O\left(\frac{x^3}{\sqrt{t}}\right)+O(x^3e^{-At}) = O\left(\frac{x^3}{\sqrt{t}}\right).$$

For $x \leqslant 1$, $t \leqslant 1$ it is

$$\sum^{\infty} O\left\{x^3t^3\left(\frac{e}{2^n}\right)^{2n}\right\} = O(x^3t^3).$$

Theorem 159. *Let $f(x)$ be an odd function of x, and let $f(x)/(1+|x|)^{\frac{1}{2}}$ belong to $L(-\infty,\infty)$. Then a necessary and sufficient condition that $f(x)$ should be expressible by the Neumann series* (11.22.1), *with the coefficients* (11.22.3), *is that $f(x)$ should satisfy* (11.22.4).

Suppose first that $f(x)$ is expressible by the above series.

It follows from the lemma that, for a fixed x,

$$\sum_{0}^{\infty} (4n+2)|J_{2n+1}(x)J_{2n+1}(t)| = O\{\min(1,t^{-\frac{1}{2}})\},$$

and the inversion of the above summation and integration is justified by absolute convergence.

It is also clear from (11.23.3) that, if $f(x)$ is expressible by (11.22.1), it is (like the sum of a power series) differentiable any number of times within the range of convergence of the series (here 0 to ∞). The final integration by parts is therefore justified. Hence Kapteyn's equation holds.

Conversely, if (11.22.4) holds, then $f'(u)$ is continuous, by the uniform convergence of the integral. The argument can therefore be reversed.

11.24. Solution of Kapteyn's equation. Since, in § 11.22, $f(x)$ is odd, (11.22.4) may be written

$$f'(u) = \tfrac{1}{2}\int\limits_{0}^{\infty}\{f(u+\xi)-f(u-\xi)\}\frac{J_1(\xi)}{\xi}\,d\xi, \tag{11.24.1}$$

or

$$f'(x) = -\tfrac{1}{2}\int\limits_{-\infty}^{\infty} f(x-y)\frac{J_1(y)}{y}\operatorname{sgn} y\,dy, \tag{11.24.2}$$

and in this form $f(x)$ is not necessarily odd.

THEOREM 160. *Let $f(x)$ belong to $L^2(-\infty,\infty)$. Then a necessary and sufficient condition that* (11.24.2) *should hold for all values of x is that*

$$f(x) = \int\limits_{-1}^{1}\phi(u)e^{-ixu}\,du,$$

where $\phi(u)$ belongs to $L^2(-1,1)$.

The Fourier transform of $g(x) = x^{-1}J_1(x)\operatorname{sgn} x$ is

$$G(x) = i\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{0}^{\infty}\frac{J_1(y)}{y}\sin xy\,dy$$

$$= i\sqrt{\left(\frac{2}{\pi}\right)}x \quad (|x| \leqslant 1), \qquad i\sqrt{\left(\frac{2}{\pi}\right)}\frac{\operatorname{sgn} x}{|x|+\sqrt{(x^2-1)}} \quad (|x| > 1).$$

Hence, if F is the transform of f,

$$\int\limits_{-\infty}^{\infty} f(x-y)\frac{J_1(y)}{y}\operatorname{sgn} y\,dy = -i\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{-\infty}^{-1}\frac{F(t)e^{-ixt}}{|t|+\sqrt{(t^2-1)}}\,dt +$$

$$+i\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{-1}^{1} tF(t)e^{-ixt}\,dt + i\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_{1}^{\infty}\frac{F(t)e^{-ixt}}{t+\sqrt{(t^2-1)}}\,dt;$$

$-\frac{1}{2}$ of the integral of this with respect to x is

$$-\frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{-1}\frac{F(t)e^{-ixt}}{|t|+\sqrt{(t^2-1)}}\frac{dt}{t} +$$

$$+\frac{1}{\sqrt{(2\pi)}}\int\limits_{-1}^{1} F(t)e^{-ixt}\,dt + \frac{1}{\sqrt{(2\pi)}}\int\limits_{1}^{\infty}\frac{F(t)e^{-ixt}}{t+\sqrt{(t^2-1)}}\frac{dt}{t},$$

and, by the theory of § 3, the necessary and sufficient condition that this should equal $f(x)$, or differ from it by a constant, is that $F(t) \equiv 0$ for $|t| > 1$. This proves the theorem.

Theorem 161. *If*

$$f(x) = f(0) + x\int_{-1}^{1} \chi(u)e^{-ixu}\,du,$$

where $\chi(u)/\sqrt{(1-u^2)}$ *belongs to* $L(-1,1)$, *then* $f(x)$ *is a solution of* (11.24.1).

The term $f(0)$ is a solution, so may be omitted. We then have

$$\frac{1}{2}\int_0^\infty \{f(x+\xi)-f(x-\xi)\}\frac{J_1(\xi)}{\xi}\,d\xi$$

$$= \frac{1}{2}\int_0^\infty \frac{J_1(\xi)}{\xi}(x+\xi)\,d\xi \int_{-1}^{1} \chi(u)e^{-i(x+\xi)u}\,du -$$

$$-\frac{1}{2}\int_0^\infty \frac{J_1(\xi)}{\xi}(x-\xi)\,d\xi \int_{-1}^{1} \chi(u)e^{-i(x-\xi)u}\,du$$

$$= -ix\int_{-1}^{1} \chi(u)e^{-ixu}\,du \int_0^\infty \frac{J_1(\xi)}{\xi}\sin\xi u\,d\xi +$$

$$+\int_{-1}^{1} \chi(u)e^{-ixu}\,du \int_0^\infty J_1(\xi)\cos\xi u\,d\xi$$

$$= -ix\int_{-1}^{1} \chi(u)u\,e^{-ixu}\,du + \int_{-1}^{1} \chi(u)e^{-ixu}\,du = f'(x)$$

(Watson, § 13.42) if the inversions are justified.

The repeated integral with the factor x outside is absolutely convergent; the inversion of the other is justified by dominated convergence provided that

$$\left|\int_0^T J_1(\xi)\cos\xi u\,d\xi\right| < \frac{K}{\sqrt{(1-u^2)}}$$

for all T.

Here the leading term in the asymptotic expansion gives terms like

$$\int_1^T \frac{\cos\xi\cos\xi u}{\sqrt{\xi}}\,d\xi = \int_1^T \frac{\cos\xi(1-u)}{\sqrt{\xi}}\,d\xi + \ldots = O\left\{\frac{1}{\sqrt{(1-u)}}\right\} + \ldots,$$

and the result follows.

$f(x) = \sin x$ is an example of this theorem.

Theorem 162. *Let $f(x)/(1+|x|)$ belong to $L^2(-\infty,\infty)$, and $f(x)/(1+|x|)^{\frac{3}{2}}$ to $L(-\infty,\infty)$. Then, if $f(x)$ satisfies Kapteyn's equation,*

$$f(x) = x\int_{-1}^{1} \chi(u)e^{-ixu}\,du,$$

where $\chi(u)$ belongs to $L^2(-1,1)$.

The formal argument here is that, if $f(x)$ satisfies Kapteyn's equation, it is expansible in the form (11.22.1); and then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{\sin(\xi-x)}{\xi-x}\,dx = \sum_{n=0}^{\infty}\frac{a_{2n+1}}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\xi-x)}{\xi-x}J_{2n+1}(x)\,dx$$

$$= \sum_{n=0}^{\infty} a_{2n+1}J_{2n+1}(\xi) = f(\xi).$$

Thus $f(x)$ satisfies Bateman's equation (with $a = 1$), and so is equal to a Fourier integral with limits $(-1, 1)$. Owing to convergence difficulties we have to apply the argument indirectly. We have instead

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)-a_1J_1(x)}{x^3}\frac{\sin(\xi-x)}{\xi-x}\,dx = \sum_{n=1}^{\infty}\frac{a_{2n+1}}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\xi-x)}{\xi-x}\frac{J_{2n+1}(x)}{x^3}\,dx$$

$$= \sum_{n=1}^{\infty} a_{2n+1}\frac{J_{2n+1}(\xi)}{\xi^3} = \frac{f(\xi)-a_1J_1(\xi)}{\xi^3}.$$

This inversion is justified by absolute convergence, since the lemma of § 11.23 shows that

$$\sum_{n=1}^{\infty}(4n+2)\int_{0}^{\infty}\left|\frac{f(t)}{t}J_{2n+1}(t)\right|dt\int_{-\infty}^{\infty}\left|\frac{\sin(\xi-x)}{\xi-x}\right|\left|\frac{J_{2n+1}(x)}{x^3}\right|dx$$

is convergent.

It now follows from Theorem 156 that

$$\frac{f(x)-a_1J_1(x)}{x^3} = \int_{-1}^{1}\phi(u)e^{-ixu}\,du,$$

where $\phi(u)$ belongs to $L^2(-1,1)$. Hence

$$\phi(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(x)-a_1J_1(x)}{x^3}e^{ixu}\,dx \quad (-1<u<1),$$

and, since $\{f(x)-a_1 J_1(x)\}/x$ belongs to $L^2(-\infty,\infty)$, $\phi(u)$ is the integral of the integral of a function of L^2. Integrating by parts twice, we obtain

$$\frac{f(x)-a_1 J_1(x)}{x} = x(ae^{ix}+be^{-ix})+(ce^{ix}+de^{-ix})+\int\limits_{-1}^{1} \chi(u)e^{-ixu}\,du,$$

where χ is $L^2(-1,1)$; since the left-hand side is $L^2(-\infty,\infty)$, a, b, c, and d must vanish. This proves the theorem.

11.25. A differential equation of fractional order. The integral equation†

$$f(x) = \frac{\lambda}{\Gamma(\alpha)}\int\limits_{x}^{\infty} (y-x)^{\alpha-1}f(y)\,dy \tag{11.25.1}$$

may be regarded as a differential equation of order α. Suppose, for example, that α is a positive integer p, that $f(x)$ tends to 0, as $x \to \infty$, with sufficient rapidity, and that

$$f_1(x) = \int\limits_{x}^{\infty} f(y)\,dy, \qquad f_2(x) = \int\limits_{x}^{\infty} f_1(y)\,dy, \ldots .$$

Then, if we integrate repeatedly by parts, and write z for $f_p(x)$, (11.25.1) becomes

$$\frac{d^p z}{dx^p} = (-1)^p \lambda z.$$

The only solutions of this are finite combinations of exponentials.

The general equation (11.25.1) is of the form (11.2.1), with

$$k(x) = \frac{\lambda x^{\alpha-1}}{\Gamma(\alpha)} \quad (x > 0), \qquad 0 \quad (x < 0).$$

The theory of § 11.2 is not applicable, since $k(x)$ does not satisfy (11.2.3). But the equation still has exponential solutions. The conditions that $f(x) = e^{-ax}$ should be a solution are that $\mathbf{R}(a) > 0$ and $\lambda = a^\alpha$, where a^α means $e^{\alpha \log a}$, and $\log a$ has its principal value. If $\lambda > 0$, a may have any of the values

$$\lambda^{1/\alpha} e^{2r\pi i/\alpha} \quad (r = 0, \pm 1, \ldots)$$

for which $|2r\pi/\alpha| < \frac{1}{2}\pi$. If $\alpha \leqslant 4$, and in particular if $\alpha < 1$, the only admissible value of a is $\lambda^{1/\alpha}$. We shall prove that in this case, at any rate, the solution is unique.

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† Hardy and Titchmarsh (7).

THEOREM 163. *Let $f(x)$ be integrable over any finite interval $0 < x_1 \leqslant x \leqslant x_2$, let $\lambda > 0$, $0 < \alpha < 1$, and let*

$$f(x) = \frac{\lambda}{\Gamma(\alpha)} \int_{\to x}^{\to\infty} (y-x)^{\alpha-1} f(y)\, dy \tag{11.25.2}$$

for every positive x. Then

$$f(x) = Ce^{-ax},$$

where $a = \lambda^{1/\alpha}$, and C is a constant.

If $\mathfrak{F}(s)$ is the Mellin transform of $f(x)$, we have formally

$$\begin{aligned}\mathfrak{F}(s) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^\infty x^{s-1}\, dx \int_x^\infty (y-x)^{\alpha-1} f(y)\, dy \\ &= \frac{\lambda}{\Gamma(\alpha)} \int_0^\infty f(y)\, dy \int_0^y x^{s-1}(y-x)^{\alpha-1}\, dx \\ &= \frac{\lambda\Gamma(s)}{\Gamma(s+\alpha)} \mathfrak{F}(s+\alpha).\end{aligned}$$

We shall prove that this is in fact true, and base our solution on it; but we cannot justify the inversions as they stand, and we have to proceed indirectly. We require the following lemma.

Let† $$f_\alpha^*(x) = \frac{1}{\Gamma(\alpha)} \int_{\to x}^{c} f(y)(y-x)^{\alpha-1}\, dy$$

for every positive x. Then, if $\beta > 0$,

$$\frac{1}{\Gamma(\beta)} \int_{\to\xi}^{c} f_\alpha^*(x)(x-\xi)^{\beta-1}\, dx = \frac{1}{\Gamma(\alpha+\beta)} \int_{\to\xi}^{c} f(y)(y-\xi)^{\alpha+\beta-1}\, dy,$$

i.e.
$$(f_\alpha^*)_\beta^* = f_{\alpha+\beta}^*. \tag{11.25.3}$$

To prove this, we have to justify the inversion

$$\int_{\to\xi}^{c} (x-\xi)^{\beta-1}\, dx \int_{\to x}^{c} f(y)(y-x)^{\alpha-1}\, dy = \int_{\to\xi}^{c} f(y)\, dy \int_{\xi}^{y} (x-\xi)^{\beta-1}(y-x)^{\alpha-1}\, dx.$$

Clearly
$$\int_{\xi+\delta}^{c} \dots \int_{\to x}^{c} \dots = \int_{\xi+\delta}^{c} \dots \int_{\xi+\delta}^{y} \dots,$$

and it is sufficient to prove that

$$I = \int_{\to\xi}^{\xi+\delta} f(y)\, dy \int_{\xi}^{y} (x-\xi)^{\beta-1}(y-x)^{\alpha-1}\, dx \to 0$$

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† See Bosanquet (1) for a proof under much more general conditions.

and $$J = \int_{\xi+\delta}^{c} f(y)\,dy \int_{\xi}^{\xi+\delta} (x-\xi)^{\beta-1}(y-x)^{\alpha-1}\,dx \to 0$$

as $\delta \to 0$. Now

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_{\to\xi}^{\xi+\delta} f(y)(y-\xi)^{\alpha+\beta-1}\,dy,$$

and the result for this follows from the second mean-value theorem and the existence of $\int_{\to\xi} f(y)(y-\xi)^{\alpha-1}\,dy$.

Also

$$J = \int_{\xi+\delta}^{c} f(y)(y-\xi)^{\alpha-1}\,dy \int_{\xi}^{\xi+\delta} (x-\xi)^{\beta-1}\left(\frac{y-x}{y-\xi}\right)^{\alpha-1} dx,$$

and the inner integral is steadily increasing with y, and its value when $y = c$ is $O(\delta^{\beta})$. Hence the result for this part also follows from the second mean-value theorem.

Proof of Theorem 163. Let $c > 1$, $0 < x \leqslant \frac{1}{2}c$, and write

$$f(x) = \frac{\lambda}{\Gamma(\alpha)} \int_{\to x}^{c} f(y)(y-x)^{\alpha-1}\,dy + \frac{\lambda}{\Gamma(\alpha)} \int_{c}^{\to\infty} f(y)(y-x)^{\alpha-1}\,dy$$
$$= \lambda f_{\alpha}^{*}(x) + \lambda g(x),$$

say. Then

$$f_{\alpha}^{*}(x) = \frac{\lambda}{\Gamma(\alpha)} \int_{\to x}^{c} \{f_{\alpha}^{*}(y)+g(y)\}(y-x)^{\alpha-1}\,dy$$
$$= \lambda f_{2\alpha}^{*}(x) + \lambda g_{\alpha}^{*}(x)$$

by (11.25.3). Hence

$$f(x) = \lambda g(x) + \lambda^{2} g_{\alpha}^{*}(x) + \lambda^{2} f_{2\alpha}^{*}(x).$$

Repeating the argument, we obtain

$$f(x) = \lambda g(x) + \lambda^{2} g_{\alpha}^{*}(x) + \ldots + \lambda^{n} g_{(n-1)\alpha}^{*}(x) + \lambda^{n} f_{n\alpha}^{*}(x), \tag{11.25.4}$$

where $$f_{n\alpha}^{*}(x) = \frac{1}{\Gamma(n\alpha)} \int_{x}^{c} f(y)(y-x)^{n\alpha-1}\,dy.$$

By taking n large enough, in particular $n\alpha > 1$, we obtain

$$|f(x)| \leqslant \phi(x) + \frac{1}{2c} \int_{x}^{c} |f(y)|\,dy, \tag{11.25.5}$$

where $\phi(x)$ is bounded as $x \to 0$. Hence $f(x)$ is bounded as $x \to 0$; otherwise there would be a sequence of values of x such that

$|f(x)| \geqslant f(y)$ $(x \leqslant y \leqslant c)$, $|f(x)| \to \infty$, which is inconsistent with (11.25.5). It then follows that $f^*_{n\alpha}(x)$ is continuous for $0 \leqslant x \leqslant c$, and hence so is $f(x)$. Denote its limit as $x \to 0$ by $f(0)$.

We can also differentiate (11.25.4), and it follows that $f'(x)$ is bounded near the origin. The argument could be carried on indefinitely, but all we require is that

$$f(x)-f(0) = O(x)$$

as $x \to 0$.

Now
$$\mathfrak{F}(s) = \int_0^1 \{f(x)-f(0)\}x^{s-1}\,dx + \int_1^{\rightarrow\infty} f(x)x^{s-1}\,dx + \frac{f(0)}{s},$$

primarily for $0 < \sigma < \alpha$, and then, as an analytic continuation of $\mathfrak{F}(s)$, for $-1 < \sigma < \alpha$. Since

$$\int_1^{\infty} f(0)x^{s-1}\,dx = -\frac{f(0)}{s} \quad (\sigma < 0),$$

we have
$$\mathfrak{F}(s) = \int_0^{\rightarrow\infty} \{f(x)-f(0)\}x^{s-1}\,dx \quad (-1 < \sigma < 0).$$

Inserting values of $f(x)$ and $f(0)$ given by (11.25.2), we obtain formally

$$\begin{aligned}
\mathfrak{F}(s) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\rightarrow\infty} x^{s-1}\,dx \int_x^{\rightarrow\infty} \{(y-x)^{\alpha-1}-y^{\alpha-1}\}f(y)\,dy + \\
&\qquad\qquad + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\rightarrow\infty} x^{s-1}\,dx \int_0^x y^{\alpha-1}f(y)\,dy \\
&= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\rightarrow\infty} f(y)\,dy \int_0^y x^{s-1}\{(y-x)^{\alpha-1}-y^{\alpha-1}\}\,dx + \\
&\qquad\qquad + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\rightarrow\infty} y^{\alpha-1}f(y)\,dy \int_y^{\rightarrow\infty} x^{s-1}\,dx \\
&= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\rightarrow\infty} f(y)\left\{\frac{\Gamma(s)\Gamma(\alpha)}{\Gamma(s+\alpha)}y^{s+\alpha-1} - \frac{y^{s+\alpha-1}}{s}\right\}dy + \\
&\qquad\qquad + \frac{\lambda}{s\Gamma(\alpha)} \int_0^{\rightarrow\infty} f(y)y^{s+\alpha-1}\,dy \\
&= \frac{\lambda\Gamma(s)}{\Gamma(s+\alpha)} \int_0^{\rightarrow\infty} f(y)y^{s+\alpha-1}\,dy = \frac{\lambda\Gamma(s)}{\Gamma(s+\alpha)}\mathfrak{F}(s+\alpha).
\end{aligned}$$

We shall show that this process is valid if $-\alpha < \sigma < 0$.

For the first term, the integral over $y \leqslant N$ can plainly be inverted, and it is sufficient to prove that

$$\int_0^N x^{s-1}\,dx \int_N^{\to\infty} \{(y-x)^{\alpha-1}-y^{\alpha-1}\}f(y)\,dy \tag{11.25.6}$$

and

$$\int_N^{\to\infty} x^{s-1}\,dx \int_x^{\to\infty} \{(y-x)^{\alpha-1}-y^{\alpha-1}\}f(y)\,dy \tag{11.25.7}$$

tend to 0 as $N \to \infty$. Now by the second mean-value theorem

$$\int_N^{\to\infty} \{(y-x)^{\alpha-1}-y^{\alpha-1}\}f(y)\,dy = \left\{\left(1-\frac{x}{N}\right)^{\alpha-1}-1\right\} \int_N^{\xi} y^{\alpha-1}f(y)\,dy;$$

the last integral is bounded, and

$$\int_0^N x^{\sigma-1}\left\{\left(1-\frac{x}{N}\right)^{\alpha-1}-1\right\} dx = N^{\sigma} \int_0^1 \{(1-u)^{\alpha-1}-1\}u^{\sigma-1}\,du = O(N^{\sigma}),$$

giving the result for (11.25.6). Also (11.25.7) is

$$\frac{\Gamma(\alpha)}{\lambda} \int_N^{\to\infty} x^{s-1}f(x)\,dx - \int_N^{\infty} x^{s-1}\,dx \int_x^{\to\infty} y^{\alpha-1}f(y)\,dy,$$

which plainly tends to 0.

The inversion of the second term is equivalent to integration by parts:

$$\int_0^{\to\infty} x^{s-1}\,dx \int_0^x y^{\alpha-1}f(y)\,dy = \left[\frac{x^s}{s}\int_0^x y^{\alpha-1}f(y)\,dy\right]_0^{\infty} - \frac{1}{s}\int_0^{\to\infty} x^{s+\alpha-1}f(x)\,dx,$$

and the integrated term tends to 0 at each limit.

Let
$$\chi(s) = \lambda^{s/\alpha}\frac{\mathfrak{F}(s)}{\Gamma(s)}.$$

Then the above result is equivalent to

$$\chi(s+\alpha) = \chi(s).$$

Thus $\chi(s)$ has the period α, and is regular for $-1 < \sigma < 0$, and so everywhere. Also, if $h(x) = \int_x^{\to\infty} f(y)y^{\alpha-1}\,dy = o(1)$,

$$\begin{aligned}\mathfrak{F}(s) &= O(1)+\int_1^{\to\infty} f(x)x^{s-1}\,dx \\ &= O(1)+h(1)+(s-\alpha)\int_1^{\infty} h(x)x^{s-\alpha-1}\,dx \\ &= O(|t|) \quad (-1 < \sigma < 0).\end{aligned}$$

Hence
$$\chi(s) = O(|t|^{A}e^{\frac{1}{2}\pi|t|})$$
for $-1 < \sigma < 0$, and so on any line parallel to the imaginary axis. Hence
$$\chi\left(\frac{\alpha \log z}{2\pi i}\right)$$
is one-valued, and $O(\log^{A} r\, r^{\frac{1}{2}\alpha})$ as $|z| = r \to \infty$, and $O(\log^{A}(1/r)r^{-\frac{1}{2}\alpha})$ as $r \to 0$. Hence $\chi(s)$ is a constant,
$$\mathfrak{F}(s) = C\Gamma(s)\lambda^{-s/\alpha},$$
and, by Theorem 32,
$$f(x) = \frac{C}{2\pi i}\int\limits_{k-i\infty}^{k+i\infty} \Gamma(s)\lambda^{-s/\alpha}x^{-s}\, ds \quad (0 < k < \alpha)$$
$$= Ce^{-x\lambda^{1/\alpha}}$$

11.26. A probability problem.† A function $f(x)$, such that $f(x) \geqslant 0$ and
$$\int\limits_{-\infty}^{\infty} f(x)\, dx = 1,$$
defines a *law of errors*, which asserts that the probability that the error in making a certain measurement lies within the range (x_1, x_2) is $\int\limits_{x_1}^{x_2} f(x)\, dx$; or that, for small δx, the probability that the error lies in $(x, x+\delta x)$ is to a first approximation $f(x)\,\delta x$.

Suppose that we observe two quantities P and Q, and that the errors in observing them, p and q, are distributed according to laws $f(x)$ and $g(x)$. It is required to find the corresponding law for $P+Q$.

If p and q are capable of taking integer values only, and the proportion of times that p is x is $f(x)$, and that q is y is $g(y)$, then the proportion of times that $p+q$ is ξ is
$$\sum_{x+y=\xi}\sum f(x)g(y) = \sum_{x} f(x)g(\xi-x),$$
i.e. the 'resultant' of f and g.

In the continuous case, a similar argument with $f(x)\,\delta x$ and $g(y)\,\delta y$ leads to
$$\int\limits_{-\infty}^{\infty} f(x)g(\xi-x)\, dx$$

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† Pólya (2).

as the law for $p+q$. We can prove this rigorously as follows. Strictly, p and q run through sets of points E_1 and E_2 such that

$$mE_1(p \leqslant x) = \int_{-\infty}^{x} f(u)\,du, \qquad mE_2(q \leqslant x) = \int_{-\infty}^{x} g(u)\,du.$$

Let
$$f_1(x) = \int_{-\infty}^{x} f(u)\,du, \qquad g_1(x) = \int_{-\infty}^{x} g(u)\,du.$$

Consider the sum

$$S = \sum_{n=-\infty}^{\infty} g_1(\xi-n\delta)[f_1\{(n+1)\delta\}-f_1(n\delta)].$$

The term in n represents the probability that p is in $(n\delta, (n+1)\delta)$ and q is $\leqslant \xi-n\delta$. For such p and q, $p+q \leqslant \xi+\delta$. On the other hand, if $p+q \leqslant \xi$, then $n\delta \leqslant p \leqslant (n+1)\delta$ and $q \leqslant \xi-n\delta$ for some n. Hence
$$mE(p+q \leqslant \xi) \leqslant S \leqslant mE(p+q \leqslant \xi+\delta).$$

Since f is L, and g_1 is continuous and tends to 0 as $x \to -\infty$ and to 1 as $x \to \infty$, it is easily seen that

$$\lim_{\delta\to 0} \sum_{n=-\infty}^{\infty} \int_{n\delta}^{(n+1)\delta} f(t)\{g_1(\xi-t)-g_1(\xi-n\delta)\}\,dt = 0,$$

i.e. that
$$\lim_{\delta\to 0} S = \int_{-\infty}^{\infty} f(t)g_1(\xi-t)\,dt.$$

Hence

$$\begin{aligned} mE(p+q \leqslant \xi) &= \int_{-\infty}^{\infty} f(t)g_1(\xi-t)\,dt \\ &= \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\xi-t} g(x)\,dx = \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\xi} g(u-t)\,du \\ &= \int_{-\infty}^{\xi} du \int_{-\infty}^{\infty} f(t)g(u-t)\,dt, \end{aligned}$$

which is equivalent to the result stated.

If $f(x)$ gives a law of errors, so does $\dfrac{1}{a}f\left(\dfrac{x}{a}\right)$. We now ask for what law of errors the resultant of two laws of this form is also of the same form.

THEOREM 164. *Let $f(x) \geqslant 0$, let $f(x)$ and $x^2f(x)$ belong to $L(-\infty,\infty)$, and let*

$$\frac{1}{c}f\left(\frac{x}{c}\right) = \frac{1}{ab}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)f\left(\frac{x-y}{b}\right)dy \tag{11.26.1}$$

for every x, where a, b, c, are given positive numbers.

Then the conditions are consistent only if $c^2 = a^2+b^2$; *and in that case*

$$f(x) = \frac{1}{\sqrt{(2\pi k)}}e^{-\frac{1}{2}x^2/k}$$

almost everywhere, where k *is a constant.*

The integral

$$\int\limits_{-\infty}^{\infty} x^m f(x)\,dx = K_m$$

exists for $m = 0, 1, 2$. Now, for $m = 0, 1, 2$,

$$\frac{1}{c}\int\limits_{-\infty}^{\infty} x^m f\left(\frac{x}{c}\right)dx = \frac{1}{ab}\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} x^m f\left(\frac{y}{a}\right) f\left(\frac{x-y}{b}\right) dy$$

$$= \frac{1}{ab}\int\limits_{-\infty}^{\infty} f\left(\frac{y}{a}\right) dy \int\limits_{-\infty}^{\infty} (y+u)^m f\left(\frac{u}{b}\right) du,$$

and, in the three cases $m = 0, 1, 2$ this gives

$$K_0 = K_0^2, \tag{11.26.2}$$

$$cK_1 = aK_1 K_0 + bK_0 K_1, \tag{11.26.3}$$

$$c^2K_2 = a^2K_2 K_0 + 2abK_1^2 + b^2K_0 K_2. \tag{11.26.4}$$

Assuming that $f(x)$ is not null, (11.26.2) gives $K_0 = 1$. Hence (11.26.3) gives

$$(a+b-c)K_1 = 0, \tag{11.26.5}$$

and (11.26.4) gives

$$(c^2-a^2-b^2)K_2 = 2abK_1^2. \tag{11.26.6}$$

But by Schwarz's inequality

$$K_1^2 < K_0 K_2 = K_2,$$

so that (11.26.6) gives

$$c^2-a^2-b^2 < 2ab,$$

$$c < a+b.$$

Hence (11.26.5) gives $K_1 = 0$, and (11.26.6) gives

$$c^2 = a^2+b^2.$$

Let

$$\frac{a}{c} = \alpha, \qquad \frac{b}{c} = \beta.$$

Then, putting $x = c\xi$, $y = c\eta$, (11.26.1) becomes

$$f(\xi) = \frac{1}{\alpha\beta}\int\limits_{-\infty}^{\infty} f\left(\frac{\eta}{\alpha}\right) f\left(\frac{\xi-\eta}{\beta}\right) d\eta, \tag{11.26.7}$$

where

$$\alpha^2+\beta^2 = 1. \tag{11.26.8}$$

Let $$\Phi(x) = \sqrt{(2\pi)}F(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}\,dt.$$

Then, by Theorem 41, (11.26.7) gives

$$\Phi(x) = \Phi(\alpha x)\Phi(\beta x). \tag{11.26.9}$$

Using (11.26.9) for each term on the right, we obtain

$$\Phi(x) = \Phi(\alpha^2 x)\Phi(\beta\alpha x)\Phi(\alpha\beta x)\Phi(\beta^2 x),$$

and so generally

$$\Phi(x) = \Phi(\gamma_{m,1}x)\Phi(\gamma_{m,2}x)\ldots\Phi(\gamma_{m,m}x),$$

where $m = 2^n$, and the m numbers $\gamma_{m,1},\ldots, \gamma_{m,m}$ are the 2^n terms obtained by expanding $$(\alpha+\beta)^n.$$

Hence $$\gamma_{m,1}^2+\gamma_{m,2}^2+\cdots+\gamma_{m,m}^2 = (\alpha^2+\beta^2)^n = 1.$$

Also $\gamma_{m,\mu}$ is of the form $\alpha^p\beta^q$, where $p+q = n$; and hence, supposing $\alpha \geqslant \beta$, we have $$\gamma_{m,\mu} \leqslant \alpha^n \quad (\mu = 1, 2,\ldots, m).$$

Hence $\max\limits_{\mu}\gamma_{m,\mu} \to 0$ as $m \to \infty$.

Now since $f(x)$, $xf(x)$, and $x^2f(x)$ belong to L, $\Phi(x)$, $\Phi'(x)$, and $\Phi''(x)$ are continuous; and

$$\Phi(0) = \int_{-\infty}^{\infty} f(t)\,dt = K_0 = 1,$$

$$\Phi'(0) = \int_{-\infty}^{\infty} f(t)it\,dt = iK_1 = 0,$$

and $$\Phi''(0) = -\int_{-\infty}^{\infty} t^2 f(t)\,dt = -k,$$

say. Hence, in the neighbourhood of $x = 0$,

$$\log\Phi(x) = u(x)+iv(x),$$

where $u(x)$ and $v(x)$ are continuous, and

$$u(0) = v(0) = u'(0) = v'(0) = 0,$$

and $$u''(0) = -k, \qquad v''(0) = 0.$$

Hence

$$\log\Phi(x) = \sum_{\mu=1}^{m}\{u(\gamma_{m,\mu}x)+iv(\gamma_{m,\mu}x)\}$$
$$= \tfrac{1}{2}x^2\sum_{\mu=1}^{m}\gamma_{m,\mu}^2\{u''(\theta_{m,\mu}\gamma_{m,\mu}x)+iv''(\theta'_{m,\mu}\gamma_{m,\mu}x)\},$$

where $0 < \theta_{m,\mu} < 1$, $0 < \theta'_{m,\mu} < 1$. As $m \to \infty$, each $\gamma_{m,\mu} \to 0$, and hence so does each $\theta_{m,\mu}\gamma_{m,\mu}$, and uniformly with respect to μ.

Hence

$$\log\Phi(x) = \tfrac{1}{2}x^2\sum_{\mu=1}^{m}\gamma_{m,\mu}^2\{-k+o(1)\}$$
$$= -\tfrac{1}{2}kx^2+o(1),$$

i.e. $$\log\Phi(x) = -\tfrac{1}{2}kx^2, \qquad \Phi(x) = e^{-\frac{1}{2}kx^2},$$

and so (by Theorem 27) almost everywhere

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}kx^2-ixt}\,dt = \frac{1}{\sqrt{(2\pi k)}}e^{-\frac{1}{2}x^2/k},$$

i.e. the law is Gauss's law.

11.27. A problem in statistical dynamics.† Consider an assemblage of atoms moving in one dimension in such a way that the fraction of them with velocities between v and $v+\delta v$ is $\int\limits_{v}^{v+\delta v} f(x)\,dx$. At a subsequent instant let a fraction $\int\limits_{w}^{w+\delta w} \phi(v,x)\,dx$ of those with velocity v have acquired increments of velocity between w and $w+\delta w$. Then, by an argument similar to that used in the previous section, the fraction of the whole which finish with velocities between v' and $v'+\delta v'$ is $\int\limits_{v'}^{v'+\delta v'} g(x)\,dx$, where

$$g(v') = \int\limits_{-\infty}^{\infty} f(v)\phi(v,v'-v)\,dv. \tag{11.27.1}$$

For a steady state $g \equiv f$, so that f satisfies the integral equation

$$f(v') = \int\limits_{-\infty}^{\infty} f(v)\phi(v,v'-v)\,dv. \tag{11.27.2}$$

Suppose now that the motion is defined as follows. The centre of mass of atoms moving with velocity v moves according to the equation

$$\frac{dV}{dt} = -\lambda V \quad (\lambda > 0),$$

so that after time t its velocity is $ve^{-\lambda t}$. Superimposed on this motion, the atoms are given increments of velocity u in time t, the proportion of those with increment between u and $u+\delta u$ being $\psi(u)\,\delta u$; and this increment is uncorrelated with v, so that the proportion of those with velocity v having the additional increment between u and $u+\delta u$ is also $\psi(u)\delta u$. It follows that

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† E. A. Milne (2); Fowler, *Statistical Mechanics*, § 19.5. Milne's original method requires heavier restrictions than those assumed here.

$$\phi(v, ve^{-\lambda t}-v+u) = \psi(u),$$

where $\psi(u)$ is independent of v, but of course depends on t. Putting $v' = ve^{-\lambda t}+u$, this is

$$\phi(v, v'-v) = \psi(v'-ve^{-\lambda t}).$$

The condition for a steady state is therefore

$$f(v') = \int\limits_{-\infty}^{\infty} f(v)\psi(v'-ve^{-\lambda t})\,dv,$$

where, as in the last section,

$$\int\limits_{-\infty}^{\infty} f(x)\,dx = 1, \qquad \int\limits_{-\infty}^{\infty} \psi(x)\,dx = 1.$$

Let F and Ψ be the transforms of f and ψ. Then

$$\begin{aligned}F(\xi) &= \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} e^{i\xi v'}\,dv' \int\limits_{-\infty}^{\infty} f(v)\psi(v'-ve^{-\lambda t})\,dv \\ &= \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} f(v)\,dv \int\limits_{-\infty}^{\infty} \psi(v'-ve^{-\lambda t})e^{i\xi v'}\,dv' \\ &= \frac{1}{\sqrt{(2\pi)}}\int\limits_{-\infty}^{\infty} f(v)\,dv \int\limits_{-\infty}^{\infty} \psi(x)e^{i\xi ve^{-\lambda t}+i\xi x}\,dx \\ &= \sqrt{(2\pi)}F(\xi e^{-\lambda t})\Psi(\xi).\end{aligned}$$

We now show that a certain assumption about the limiting behaviour of $\psi(x)$ as $t \to 0$ actually determines all the functions completely. We assume that positive and negative increments u are equally likely, so that $\psi(u) = \psi(-u)$; and also that, as $t \to 0$, for any fixed positive δ,

$$\int\limits_{0}^{\delta} x^2\psi(x)\,dx \sim at, \qquad \int\limits_{\delta}^{\infty} \psi(x)\,dx = o(t),$$

where a is a constant. It follows that, as $t \to 0$, for a fixed ξ,

$$\begin{aligned}\sqrt{(2\pi)}\Psi(\xi)-1 &= \int\limits_{-\infty}^{\infty} \psi(x)(e^{i\xi x}-1)\,dx = 2\int\limits_{0}^{\infty} \psi(x)(\cos\xi x-1)\,dx \\ &= -\xi^2\int\limits_{0}^{\delta} x^2\psi(x)\,dx + \int\limits_{0}^{\delta} \psi(x)O(x^4)\,dx + O\left(\int\limits_{\delta}^{\infty} \psi(x)\,dx\right) \\ &= -\xi^2\{at+o(t)\}+O(\delta^2 t)+o(t),\end{aligned}$$

and by choosing first δ and then t sufficiently small it follows that

$$\sqrt{(2\pi)}\Psi(\xi)-1 \sim -\xi^2 at.$$

Hence

$$\frac{F(\xi)-F(\xi e^{-\lambda t})}{\xi-\xi e^{-\lambda t}} = \frac{F(\xi e^{-\lambda t})\{\sqrt{(2\pi)}\Psi(\xi)-1\}}{\xi-\xi e^{-\lambda t}} \to -\frac{a\xi F(\xi)}{\lambda}$$

as $t \to 0$; i.e.

$$F'(\xi) = -a\xi\lambda^{-1}F(\xi).$$

Hence

$$F(\xi) = Ce^{-\frac{1}{2}a\xi^2/\lambda},$$

and $C = F(0) = (2\pi)^{-\frac{1}{2}}$. Hence

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}a\xi^2/\lambda-i\xi x}\,d\xi = \left(\frac{\lambda}{2\pi a}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\lambda x^2/a},$$

i.e. the distribution is 'Maxwellian'.

Hence also

$$\Psi(\xi) = \frac{1}{\sqrt{(2\pi)}}\frac{F(\xi)}{F(\xi e^{-\lambda t})} = \frac{1}{\sqrt{(2\pi)}}\exp\left\{-\frac{a\xi^2}{2\lambda}(1-e^{-2\lambda t})\right\},$$

and hence

$$\psi(x) = \left\{\frac{\lambda}{2\pi a(1-e^{-2\lambda t})}\right\}^{\frac{1}{2}}\exp\left\{-\frac{\lambda x^2}{2a(1-e^{-2\lambda t})}\right\}.$$

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